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XXII. *The Determination of Coma from a Central Ray.* By

T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received May 1, 1917.

The optical sine condition, due to Clausius, but more generally associated with the name of Abbe, embodies a result of great importance to the lens designer, since it enables him to form an estimate of the degree to which a system is corrected for coma * without having to calculate any rays other than those necessary to determine the spherical aberration. The extent of the departure from the sine condition is not in itself a measure of the amount of coma. Investigations of the meaning which should be given to a departure from the sine condition when spherical aberration is present have been made by Conrady † and Chalmers.‡ The conclusion reached by Conrady is that departure from the sine condition produces a proportionate zonal change of magnification. That this is true for rays lying in a principal plane is indeed suggested by the well-known law that $\mu h \sin \theta$ is invariant on refraction, where μ is the refractive index, θ the inclination of the ray to the axis, and h a short length perpendicular to the axis at the point where it is met by the ray. Such a statement of the meaning of the condition may suffice to give an indication of the extent to which it is permissible to depart from the condition, but it is evidently not a quite exact statement. For consider a system which is quite free from aberration on the axis, but which does not satisfy the sine condition. If this departure from the sine condition involved only zonal differences of magnification, the image of a point off the axis would be a straight radial line, an effect inconsistent with the essential character of coma, which in this case is the only aberration the departure from the sine condition involves.

* Coma is to be understood to include all those aberrations, the presence of which influences the fulfilment or non-fulfilment of the sine condition, excepting those which do not disappear when the object point is situated on the axis of the instrument; the latter are referred to as spherical aberrations. Generally speaking, rays from a given object point meeting the stop in an arc of a circular zone will intersect the image plane in a circular arc of equal angular extent when spherical aberration alone is present. When coma alone is present the same rays will meet the image plane in an arc of twice that angular extent.

† "The Optical Sine-Condition," Monthly Notices of R.A.S., Vol. LXV., p. 501.

‡ "The Sine Condition in Relation to the Coma of Optical Systems," Proc. Phy. Soc., Vol. XXII., p. 1.

Chalmers expresses his results in the form :—

Departure from sine condition

$=a\times$ longitudinal spherical aberration

$+b$(1/3 coma of first order $+$1/5 coma of second order

$+$ . . . .),

where a and b are constants.

He assumes that the effect of coma in the image plane may be represented by

$$\frac{\delta x}{x}=i_1(2\cos^2\psi+1)\sin^2 A+i_2(4\cos^2\psi+1)\sin^4 A+\ldots,$$

$$\frac{\delta y}{x}=i_1(2\sin\psi\cos\psi)\sin^2 A+i_2(4\sin\psi\cos\psi)\sin^4 A+\ldots,$$

where x is the distance from the axis at which a perfect image would be formed, δx and δy are the displacements in the image plane due to coma of the intersection point of a ray from a zone subtending an angle $2A$ at the axial point of the image plane, and ψ is the angle made by the axial plane through the point in which the ray meets the zone with the axial plane through the object point—*i.e.*, with the plane $y=0$. It will be shown that these assumptions involve a special interpretation of what is to be understood by the term zone.

The problem may be simply investigated by the aid of a potential function. Let the origins of co-ordinates in the object and image spaces be the point in which the axis meets the object plane, and its image formed by paraxial rays respectively. The co-ordinates and corresponding direction cosines in the object space will be denoted by x, y, z and l, m, n, and in the image space by the same letters accented. The axes of x and x' coincide with the axis of symmetry of the system. The length of the path between a given point of the object plane and the foot of the perpendicular to the refracted ray from the origin in the image space being denoted by U,* the potential function to be used will be B, where $U+B=$constant The variables being y, z, m', n', it is not difficult to show that the direction cosines of the incident ray and the point of intersection of the refracted ray with the paraxial image plane are given by

$$m=\frac{\partial B}{\partial y},\ n=\frac{\partial B}{\partial z},\ y'=\frac{\partial B}{\partial m'},\ z'=\frac{\partial B}{\partial n'},$$

* U is called by Bromwich the "modified characteristic function."

where the initial and final refractive indices are taken to be unity. The extension to the more general case will be easily obtained later.

Since the system is symmetrical about the x axis B must evidently be a function of y^2+z^2, $m'y+n'z$, and $m'^2+n'^2$ only. Taking the focal length to be unity, B may be written in the form—

$$
\begin{aligned}
B=&g(m'y+n'z)+\tfrac{1}{2}g(y^2+z^2)\\
&+\tfrac{1}{4}\alpha_1(m'^2+n'^2)^2+\tfrac{1}{6}\alpha_2(m'^2+n'^2)^3+\tfrac{1}{8}\alpha_3(m'^2+n'^2)^4+\ ..\\
&+(m'y+n'z)\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2\\
&\qquad\qquad+\beta_3(m'^2+n'^2)^3+...\}\\
&+(y^2+z^2)\{\gamma_1(m'^2+n'^2)+\gamma_2(m'^2+n'^2)^2+\gamma_3(m'^2+n'^2)^3+..\}\\
&+(m'y+n'z)^2\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+..\}\\
&+\text{ terms involving higher powers of } y \text{ and } z.
\end{aligned}
$$

The magnification of the object produced by paraxial rays is evidently g. The image point for the ray defined by $(y\ 0,\ m',\ n')$ is—

$$
\begin{aligned}
y'=&gy+m'\{\alpha_1(m'^2+n'^2)+\alpha_2(m'^2+n'^2)^2+\alpha_3(m'^2+n'^2)^3+..\}\\
&+y\{\beta_1(3m'^2+n'^2)+\beta_2(5m'^2+n'^2)(m'^2+n'^2)\\
&\qquad\qquad+\beta_3(7m'^2+n'^2)(m'^2+n'^2)^2+..\}\\
&+\text{ terms involving higher powers of } y.
\end{aligned}
$$

$$
\begin{aligned}
z'=&n'\{\alpha_1(m'^2+n'^2)+\alpha_2(m'^2+n'^2)^2+\alpha_3(m'^2+n'^2)^3+..\}\\
&+2m'n'y\{\beta_1+2\beta_2(m'^2+n'^2)+3\beta_3(m'^2+n'^2)^2+..\}\\
&+\text{ terms involving higher powers of } y.
\end{aligned}
$$

The coma terms in the expressions for y' and z' are those involving y to the first power as a factor when m' and n' assume the values they bear for a ray from the axial point of the object. Let the direction cosines for such a ray be denoted by capital letters L, M, N, L', M', N'. Although m' and n' may differ very little from M' and N', it is not permissible to replace the small letters by capitals in the equations for y' and z', for m' and n' are only actually equal to M' and N' when $y=0$. The exact expressions will be—

$$
\left.\begin{aligned}
m'&=M'+yp+y^2p'+..\\
n'&=N'+yq+y^2q'+..
\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \qquad (1)
$$

where $p, q, p', q', \ldots$ are functions of M' and N'. In the present case only p and q are involved in the determination of

coma. The substitution of $M'+yp$ and $N'+yq$ for m' and n' leads to

$$
\begin{aligned}
y' &= gy+M'\{\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2 \\
&\qquad\qquad +\alpha_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y[(\beta_1+\alpha_1 p)(3M'^2+N'^2)+(\beta_2+\alpha_2 p)(5M'^2+N'^2)(M'^2+N'^2) \\
&\qquad\qquad +(\beta_3+\alpha_3 p)(7M'^2+N'^2)(M'^2+N'^2)^2+\ldots \\
&\quad +2qM'N'\{\alpha_1+2\alpha_2(M'^2+N'^2)+3\alpha_3(M'^2+N'^2)^2+\ldots\}] \\
&\quad + \text{terms involving } y^2, \text{ \&c.,}
\end{aligned}
$$

and

$$
\begin{aligned}
z' &= N'\{\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2+\alpha_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y[2M'N'\{\beta_1+\alpha_1 p+2(\beta_2+\alpha_2 p)(M'^2+N'^2) \\
&\qquad\qquad +3(\beta_3+\alpha_3 p)(M'^2+N'^2)^2+\ldots\} \\
&\quad +q\{\alpha_1(M'^2+3N'^2)+\alpha_2(M'^2+5N'^2)(M'^2+N'^2) \\
&\qquad\qquad +\alpha_3(M'^2+7N'^2)(M'^2+N'^2)^2+\ldots\}] \\
&\quad + \text{terms involving } y^2, \text{ \&c.}
\end{aligned}
$$

Comparison with Chalmers' initial equations shows that he has assumed $p=$constant and $q=0$.

The direction cosines of the incident ray are

$$
\begin{aligned}
m &= gm'+gy+m'\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2 \\
&\qquad\qquad +\beta_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2y\{\gamma_1(m'^2+n'^2)+\gamma_2(m'^2+n'^2)^2+\gamma_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2m'^2 y\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+\ldots\}+\ldots
\end{aligned}
$$

and

$$
\begin{aligned}
n &= gn'+n'\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2+\beta_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2m'n'y\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+\ldots\}+\ldots
\end{aligned}
$$

or, substituting for m' and n' in terms of M' and N'—

$$
\begin{aligned}
m &= gM'+gy(1+p)+M'\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2 \\
&\qquad\qquad +\beta_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y\{(2\gamma_1+\beta_1 p)(M'^2+N'^2)+(2\gamma_2+\beta_2 p)(M'^2+N'^2)^2+\ldots\} \\
&\quad +2M'^2 y\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad +2M'y(M'p+N'q)\{\beta_1+2\beta_2(M'^2+N'^2) \\
&\qquad\qquad +3\beta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad + \ldots\ldots
\end{aligned}
$$

and

$$
\begin{aligned}
n &= gN'+gqy+N'\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2 \\
&\qquad\qquad +\beta_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +2M'N'y\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad +qy\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2+\ldots\} \\
&\quad +2N'(M'p+N'q)y\{\beta_1+2\beta_2(M'^2+N'^2) \\
&\qquad\qquad +3\beta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad + \ldots\ldots
\end{aligned}
$$

It will be convenient to adopt a more concise notation. Put

$$
\begin{aligned}
a&=\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2+\alpha_3(M'^2+N'^2)^3+\dots\\
a'&=2\{\alpha_1+2\alpha_2(M'^2+N'^2)+3\alpha_3(M'^2+N'^2)^2+\dots\}\\
b&=g+\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2+\beta_3(M'^2+N'^2)^3+\dots\\
b'&=2\{\beta_1+2\beta_2(M'^2+N'^2)+3\beta_3(M'^2+N'^2)^2+\dots\}\\
c&=g+2\{\gamma_1(M'^2+N'^2)+\gamma_2(M'^2+N'^2)^2+\gamma_3(M'^2+N'^2)^3+\dots\}\\
d&=2\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\dots\}
\end{aligned}
$$

so that

$$
\left.\begin{aligned}
y'&=aM'+y\{ap+b+M'(M'p+N'q)a'+M'^2b'\}+\dots\\
z'&=aN'+y\{aq+N'(M'p+N'q)a'+M'N'b'\}+\dots
\end{aligned}\right] \quad . \quad (2)
$$

$$
\left.\begin{aligned}
m&=bM'+y\{bp+M'(M'p+N'q)b'+c+M'^2d\}+\dots\\
n&=bN'+y\{bq+N'(M'p+N'q)b'+M'N'd\}+\dots
\end{aligned}\right] \quad . \quad (3)
$$

The quantities corresponding to y', z', m, n for the central ray are denoted by Y', Z', M, N. They are obtained by putting $y=0$ into equations (2) and (3). The quantity a defines the central spherical aberration and b is the sine ratio, which is constant when the sine condition is satisfied.

Equations (1), (2) and (3) show that p and q serve to define with which ray from $(y, 0)$ the central ray is compared. Suppose the two rays intersect in the object space at a point distant ρ from the origin. The conditions

$$\frac{l}{L}=\frac{m}{M-y/\rho}=\frac{n}{N}$$

are then satisfied.

The relation $\dfrac{l}{L}=\dfrac{n}{N}$ gives

$$\{p(b^3+b')+b^2c+d\}M'N'+q\{b-b^3M'^2+b'N'^2\}=0, \quad . \quad (4)$$

and $\dfrac{m}{M-y/\rho}=\dfrac{n}{N}$ reduces to

$$\frac{1}{\rho}+c=b\left(\frac{M'}{N'}q-p\right). \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, if the emergent rays meet in a point distant ρ' from the intersection of the central ray with the paraxial image plane,

$$\frac{\rho'L'}{l'}=\frac{Y'-y'+\rho'M'}{m}=\frac{Z'-z'+\rho'N'}{n'}.$$

The equality of the last two expressions yields

$$(\rho'+a')\left(\frac{M'}{N'}q-p\right)=b, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the other condition leads to

$$\left\{\frac{\rho'}{1-M'^2-N'^2}+a+(M'^2+N'^2)a'\right\}(M'p+N'q)+M'b$$
$$+M'(M'^2+N'^2)b'=0. \quad . \quad (7)$$

If both incident and emergent rays meet, equations (5) and (6) give

$$\left(\frac{1}{\rho}+c\right)(\rho'+a)=b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as the equation for finding the point of intersection of the refracted rays when that for the incident rays is known. In order that the rays should actually intersect in this way p and q must receive values which enable both the conditions (4) and (7) to be satisfied. In general, whether the rays meet or not (8) is the relation determining the position of the radial focal line for a given object point.* This result does not hold for rays lying in a primary or axial plane, for which case q should be eliminated between (4) and (5) and N' put equal to zero in (7). The corresponding equations are, therefore,

$$\frac{1-b^2M'^2}{r}+p(b+M'^2b')+c+M'^2d=0, \quad . \quad . \quad . \quad (9)$$

$$\frac{r'p}{1-M'^2}+p(a+M'^2a')+b+M'^2b'=0, \quad . \quad . \quad . \quad (10)$$

or, eliminating p,

$$\left(\frac{1-b^2M'^2}{r}+c+M'^2d\right)\left(\frac{r'}{1-M'^2}+a+M'^2a'\right)=(b+M'^2b')^2, \quad (11)$$

where r and r' have been written instead of ρ, ρ' to distinguish the different cases. Equation (11) fixes the position in which the central ray is met by the transverse focal line for a given object point.

Since from equation (4) q involves $M'N'$ as a factor, the value of q is seen from equations (2) to be quite immaterial as regards the refraction of rays associated with the ray $M'=0$. The points, therefore, of the comatic ring lying in the plane of the object point are determined by p alone. The value of q, however, does affect the shape of the ring. Equation (4) shows that the ring from a zone cannot be assumed to be circular unless the zone is so defined that $q=0$. Moreover, under these conditions p can only be a constant if certain relations hold among the coefficients of the system, and these coefficients

* This equation may be compared with equations (9) of p. 508, Proc. Phys. Soc., Vol. XXVII.

evidently involve the curvature and astigmatism in the system. It is, therefore, not permissible to assume that coma can be divided into terms of various orders which will automatically be resolved into components having displacements for rays in primary and secondary planes in the ratios 3 : 1, 5 : 1, 7 : 1, &c., by expanding in powers of the sine of the semi-angular aperture of a symmetrically placed circular zone. Chalmers' formula, though probably reliable in most cases, will consequently not be exact for large apertures. Assuming, however, a definition of a zone which leads to $q=0$, it is possible to find the two image points in the primary or axial plane, and so determine exactly the amount of the coma present.

Take first the ray in the secondary plane $M'=0$, for which

$$y'=y(ap+b),\ z'-Z'=0.$$

From equation (6)

$$ap+b=-p\rho'. \qquad \qquad (12)$$

The distances from the axis of the conjugate points on the ray are, for the object space,

$$\rho N=\rho bN',$$

and for the image space

$$(\rho'+a)N'.$$

Let the ratio of these heights be σ; then

$$\rho'+a=\rho b\sigma,$$

and by comparison with (12)

$$p\rho\sigma=-1,$$

so that

$$\frac{y'}{y}=ap+b=\frac{\rho'}{\rho\sigma}. \qquad \qquad (13)$$

Now, consider rays in the primary plane for which $N'=0$. Equations (2) give

$$z'=0,$$

and

$$y'-Y'=y\{-p+b+M'^2(a'p+b')\}$$
$$=-y\frac{r'p}{1-M'^2},$$

by equation (10).

The p in this instance is as a rule not the p of the previous case. From equations (3) and (9)

$$m-M=m-bM'$$
$$=y\{bp+M'^2b'p+c+M'^2d\}$$
$$=-y\cdot\frac{1-M^2}{r}.$$

Also from (1)

$$m'-M'=yp,$$

and, therefore,

$$\frac{m-M}{m'-M'}=-\frac{1-M^2}{pr},$$

and

$$\frac{y'-Y'}{y}=\frac{r'(1-M^2)}{r(1-M'^2)}\cdot\frac{m'-M'}{m-M},$$

$$=\frac{r'}{rs}\sqrt{\frac{1-M^2}{1-M'^2}}, \quad . \quad . \quad . \quad . \quad . \quad 14)$$

where s is the reciprocal of the angular magnification at the conjugate points determined by r and r'. If $\sin\theta$ and $\sin\theta'$ are written for M and M' respectively the conditions for freedom from coma may evidently be written,

$$\frac{\rho'}{\sigma\rho}=\frac{r'\sec\theta'}{sr\sec\theta}=\text{constant for all rays.} \quad . \quad . \quad (15)$$

These conditions hold whatever the spherical aberration on the axis may be.

The quantities

$$\frac{r'\sec\theta'}{sr\sec\theta} \text{ and } \frac{\rho'}{\sigma\rho}$$

may evidently be looked upon as the magnifications in the primary and secondary directions for small pencils, of which the ray (M', N') is a member. If these are greater than the magnification for paraxial rays the distance of the comatic ring from the axis will be greater than that of the image point for a ray through the centre of the objective. It is possible for one of these magnifications to be greater and the other less than the paraxial magnification, a case which shows the importance of finding both quantities if a correct estimate of the magnitude of the comatic patch is to be obtained.

If the external refractive indices differ it is necessary to regard g, s and σ as the reciprocals of angular magnifications. The quantities which have hitherto been called lengths will then be lengths multiplied by refractive indices. As, however, only the ratios of $\frac{y'-Y'}{y}, \frac{r'}{r}, \frac{\rho'}{\rho}$ to one another are involved in the final result, and these are all altered in the same proportion, it will be permissible to look upon the quantities that have been called lengths as absolute lengths, making g a linear magnification, while s and σ remain as before, reciprocals of angular magnifications, or linear magnifications multiplied by μ'/μ,

where μ and μ' are the refractive indices of the object and image spaces respectively.

The final conditions in their general form might well have been suggested by the theory of oblique pencils of rays. Let Fig. 1 represent an axial plane of the system OO' is the axis, and OP, OQ incident rays arising from the axial point O, which are refracted as $P'O'$, $Q'O'$. Other rays, YP and YQ, start from Y, a point in the plane of the diagram on the perpendicular to the axis through O, and after refraction emerge as $P'Y'$ and $Q'Y'$. If the distance OY is small, and P and Q are equidistant from the axis, and in the same normal plane, P' and Q' will lie on another normal plane,

Fig. 1.

and be equidistant from the axis, and Y' will lie on the perpendicular to the axis through O'. P, P' and Q, Q' are pairs of conjugate points, and provided the angles are small the ratio of the angle between the two incident rays OP, YP to the angle between the corresponding emergent rays $P'O'$, $P'Y'$ is constant. Call this ratio s, so that $\widehat{OPY} = s \,.\, \widehat{O'P'Y'}$. Let ON, $O'N'$ be perpendiculars on to the rays PY, $P'Y'$. Then $ON = OP \,.\, \widehat{OPY}$ and $O'N' = P'O' \,.\, \widehat{O'P'Y'}$, or

$$\frac{ON}{O'N'} = \frac{s \,.\, OP}{P'O'}.$$

Now, if the angles between OP and $P'O'$ and the axis are θ, θ'. the angles $\widehat{YON}$ and $\widehat{Y'O'N'}$ are also θ θ'. Thus, $ON = OY \cos\theta$, $O'N' = O'Y' \cos\theta'$, or

$$\frac{O'Y'}{OY} = \frac{P'O' \sec\theta'}{s \,.\, OP \sec\theta}.$$

If, instead of considering the length $O'Y'$ in the plane in which the central ray meets the axis another normal plane, such as that meeting the axis in I were taken, the result obtained would be—

$$\frac{JK}{OY} = \frac{P'J \sec\theta'}{s \,.\, OP \sec\theta}.$$

Figs. 2 and 3 relate to rays in a secondary plane. In Fig. 2 the rays from O and Y are seen superposed. Fig. 3 is a view in which the planes containing both central and oblique secondary rays appear as straight lines. The secondary conjugate foci on the central rays are p, p' and q q'. The point

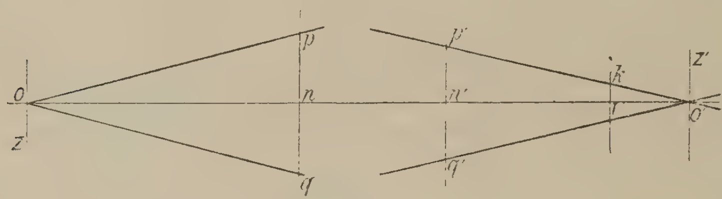

Fig. 2.

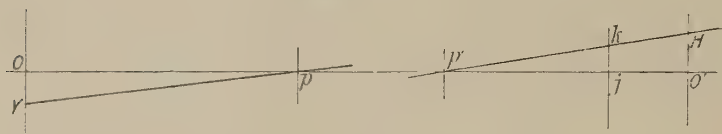

Fig. 3.

of union of the two oblique rays is H on the line from O' through Y' (Fig. 1). The ratio of the angles $\widehat{OpY}$, $\widehat{O'p'H}$ is a constant σ so long as these angles are small. Thus,

$$\frac{HO'}{OY}=\frac{p'O' . \widehat{O'p'H}}{Op . \widehat{OpY}}=\frac{p'O'}{\sigma . Op}$$

or, if measurements are to be made in the plane through I and the rays $p'O'$ and $p'H$ meet this plane in j and k respectively,

$$\frac{jk}{OY}=\frac{p'j}{\sigma . Op}$$

Now, if h is a short length on a straight line through p normal to the plane Opn, and h' is its image, and therefore a length on a straight line through p' normal to the same plane,

$$\sigma=\frac{\mu' h'}{\mu h}.$$

But the lengths h' and h can be determined by rotating the primary plane containing p through a small angle about the axis, that is to say,

$$\sigma=\frac{\mu' h'}{\mu h}=\frac{\mu' . p'n'}{\mu . pn}.$$

The length $O'Y'$ gives the distance from the axis at which the rays YP, YQ in a primary plane unite, and $O'H$ gives the distance at which the rays Yp, Yq in a secondary plane unite. When coma is present but there is no spherical aberration, as in the plane through O', the difference between $O'Y'$ and $O'H$ gives the diameter of the comatic circle. The conditions for freedom from coma are the equality of $O'Y'$ and $O'H$ to one another and to the distance from O' of the paraxial image of Y. When spherical aberration is present, as when the measurements are made in the plane through I, the light from Y through the zone $PpQq$ will meet the image plane in a curve exactly similar to that associated with the rays from O through this zone if JK and jk are equal. The condition of freedom from coma throughout the lens aperture is that for every zone JK and jk shall be equal to the displacement for paraxial rays. The analysis based on the potential function B shows that the conditions to which this geometrical treatment leads hold rigorously no matter how great the inclination of the incident beams of light may be.

In the application of the results to the determination of the amount of coma present in a system, the ray (M', N') will usually pass near the margin of the aperture. The limiting stop may be a component lens or a hole in an opaque screen placed in the object space, the image space, or in an intermediate position between some of the component lenses. One of the intersection points will usually be selected in the plane of the stop or an image position corresponding approximately to that plane. It then only remains to find the two conjugate points and the corresponding primary and secondary magnifications to determine the coma. The determination of these conjugate points and magnifications is a type of calculation with which optical computers are quite familiar, and so offers no difficulty.

The effective stop of many important systems is in the neighbourhood of the principal points. The properties usually ascribed to the principal planes will not hold for any but paraxial rays, neither in general can the planes be replaced by a pair of surfaces which stand to one another in the relation of object and image, and have for a greater number of rays the properties possessed by the principal planes for paraxial rays. For there will usually be astigmatism in the image of a surface through a principal point; in fact, the only way in which astigmatism can be removed from the image it is desired to

correct is to transfer it to the stop. The place of each principal plane must, therefore, be taken by two surfaces, one of which will relate to rays in a primary, and the other to rays in a secondary plane. The surfaces in the general case will only relate strictly to rays coming from a restricted axial region, and will alter in shape with changes in the part of the axis from which the rays considered arise, just as the real image of an object in a lens not free from spherical aberration and coma will have varying focal surfaces as the stop is moved along the axis. The results that have been proved may then be stated in the form that the coma is determined by the shape of these principal surfaces, and coma is removed from a system by ensuring that these surfaces are suitably curved. This is clearly seen in the case of any thin system of lenses where the principal surfaces are free from spherical aberration, coma and astigmatism as well as from distortion; an increase in the curvature of each lens surface by the same amount R causes an increase in the curvature of each principal surface by $R\,(1+\varpi)$ in the same sense, where ϖ is the Petzval sum. This, combined with the fact that the difference in the curvatures of the principal surfaces is ϖ times the power of the lens, accounts for the prominent part taken by the Petzval sum in the theory of the aberrations of thin lenses.*

In one particular case, which is of frequent occurrence, there is no need to make any special calculation to determine the amount of coma. This is when the object is at infinity. The values of r and ρ are then equal for all rays and the secondary second principal surface is determined by the intersection of the paths of the incident and refracted rays. The constancy of the length intercepted on the refracted ray between the incident ray produced and the image plane is the condition for freedom from coma in the secondary plane. If this condition is fulfilled for all rays it is easy to see from the equation for y' that the corresponding condition for rays in an axial plane must necessarily be satisfied also. When the condition is not satisfied the displacement due to coma of rays in a secondary plane is

$$-\left(\frac{\rho'}{f}+1\right)y_0',$$

and due to rays in a primary plane is

$$-\left(1+\tan\theta'\,\frac{d}{d\theta'}\right)\left(\frac{\rho'}{f}+1\right)y_0',$$

* See Proc. Phys. Soc., Vol. XXVII., p. 494.

where y_0' is the distance from the axis at which the image due to paraxial rays is formed, f is the focal length and ρ' is the length intercepted in the way just described on the refracted ray inclined at θ' with the axis. It is evident that when there is no spherical aberration and the condition $\rho' = f$ is satisfied the second condition also holds. In this form the conditions are of value in the calculation of telescope objectives by trigonometrical methods. Interesting alternative forms for the total primary and secondary displacements are

$$-\frac{d(\rho' \sin \theta')}{d(\sin \theta')} \cdot \frac{y_0'}{f} \text{ and } -\frac{(\rho' \sin \theta')}{\sin \theta'} \cdot \frac{y_0'}{f},$$

where it will be noted that $\rho' \sin \theta'$ is the length on the image plane intercepted between the incident and refracted rays.

The conditions under which Conrady's statement is correct may be readily found from the relations that have been proved in the course of this investigation. Equation (2) shows that any equation which involves the sine ratio, but not its differential coefficient can only refer to secondary rays, and also that the sine ratio will only measure the magnification for secondary rays if either $a=0$ or $p=0$. In the former case the zone considered is free from spherical aberration, and in the latter case the effective stop is in the object space in the surface containing the radial focal lines for an object at infinity in the image space. When the stop is so placed the primary and secondary magnifications are

$$\frac{\mu \, d(\sin \theta)}{\mu' \, d(\sin \theta')} \text{ and } \frac{\mu \sin \theta}{\mu' \sin \theta'}$$

respectively, but this expression for the primary magnification will only apply to the case when there is freedom from spherical aberration and the stop is not in the focal surface described, if the further condition is satisfied that the value of the spherical aberration is stationary on passing through this zone.

Exact formulæ of the class to which Conrady's and Chalmers' results belong may be found from the general equations already established. Such formulæ are not as general as those of equations (13) and (14), since the co-ordinates of rays in the object space do not appear in them, and the stop must consequently be assumed to lie in the image space. Let the stop lie in a plane at a distance ξ from the image plane, the equation of this

plane being therefore $x'=\xi$. Denote by g_1 and g_2 the primary and secondary magnifications of the image, so that—

$$g_1=g+\text{primary magnification due to coma,}$$

and $$g_2=g+\text{secondary magnification due to coma.}$$

Eliminate p between equations (12) and (13). Then—

$$\frac{y}{y'}\cdot b-1=\frac{a}{\rho'}, \qquad (16)$$

or, if the longitudinal spherical aberration is denoted by A—

$$\frac{1}{g_2}\cdot\frac{\mu\sin\theta}{\mu'\sin\theta'}-1=\frac{A}{\xi}, \qquad (17)$$

showing that the result $g_2=\frac{\mu\sin\theta}{\mu'\sin\theta'}$ holds when either $A=0$, or $\xi=\infty$, as has already been pointed out. Equation (17) is the form Chalmers' condition takes when the assumption that his coefficients i_1, i_2, &c., are independent of the zone of the lens is eliminated.* The corresponding relation for the primary magnification is obtaining by writing $a+M'^2a'$, $b+M'^2b'$ and $r'/(1-M'^2)$ for a, b and ρ' respectively in equation (16). The substitution of $\sin\theta'$ for M', $A\sec\theta'$ for a, $\sin\theta/\sin\theta'$ for b, and $\xi\sec\theta'$ for r', leads at once to

$$\frac{1}{g_1}\cdot\frac{\mu}{\mu'}\cdot\frac{d(\sin\theta)}{d(\sin\theta')}-1=\frac{1}{\xi}\left(A+\sin\theta'\cos\theta'\frac{dA}{d\theta'}\right), \qquad (18)$$

as the equation for the determination of g, the primary magnification. The conditions under which the primary magnification is represented by $\frac{\mu}{\mu'}\,\frac{d(\sin\theta)}{d\,(\sin\theta')}$ may be verified from this equation. The formulæ (17) and (18) will in a number of cases be more readily applied to the results obtained in tracing rays for the determination of spherical aberration than those given earlier in the Paper. They are not, however, immediately applicable to the case of an object at infinity, and are not so suggestive as are equations (13) and (14) of the means that

* In Chálmer's notation the equation would be written

$$\frac{OO_1}{JJ_1}\cdot\frac{\mu_0\sin B}{\mu_2\sin A}-1=\frac{KI}{N_1K},$$

which is simply a rearrangement of the equation

$$JJ_1=OO_1\frac{\mu_0\sin B}{\mu_2\sin A}\cdot\frac{N_1K}{N_1I},$$

he has reached on the middle of p. 6 before introducing approximations.

should be taken to remove coma in excess of the limit that can be tolerated. It is important in applying all the results to make sure that correct signs are given to all the quantities. In the case which most frequently arises—

θ, ρ, r, σ, s are positive, and

θ', ρ', r', g, g_1, g_2, ξ, x are negative.

The spherical aberration A has the sign usually adopted—viz., positive for under-correction, negative for over-correction.

It may be noted that when the stop is in the image space the comatic displacements in the image of rays through a symmetrically-placed circular zone of the stop will necessarily obey the circular law described on p. 293. For, if $P-N'^2Q$ and $M'N'Q$ are written for p and q respectively, equation (7) becomes

$$\left\{\frac{\rho'}{1-M'^2-N'^2}+a+(M'^2+N'^2)a'\right\}P+b+(M'^2+N'^2)b'=0,$$

showing that P, like all the other variables, is a function of $(M'^2+N'^2)$, and thus constant for the zone. The new form of equation (6),

$$(\rho'+a)\{(M'^2+N'^2)Q-P\}=b,$$

shows that Q also is constant for the zone. Equations (2) may now be written in the form—

$$y'-Y'=-y\rho'\left\{\frac{(1-N'^2)P}{1-M'^2-N'^2}-N'^2Q\right\}$$

$$z'-Z'=-y\rho'\left\{\frac{P}{1-M'^2-N'^2}+Q\right\}M'N',$$

which indicate displacements of the type considered. It is, therefore, unnecessary in this case to assume $q=0$. The generality of the results obtained by assuming $q=0$ is not, of course, affected since the displacements are determined by p for both $M'=0$ and $N'=0$.

If this substitution for p and q is made in (4) the result is—

$$P(b^3+b')+b^2c+d+Q\{b-(M'^2+N'^2)b^3-N'^2(b^2c+d)\}=0,$$

so that the corresponding incident rays will only intersect on a circle, if either $Q=0$ or $b^2c+d=0$. The latter alternative is evidently the condition for freedom from astigmatism in the image of the stop for pencils of rays meeting the axis near the origins of co-ordinates. It is interesting to notice that the condition for freedom from astigmatism in a zone should

contain the square of the sine ratio with respect to the centre of the stop for rays through that zone.

Equations (13) and (14) show at once why in first order coma the displacement of the intersection point of rays in a primary plane is three times that of the point for rays in a secondary plane. For it may be assumed that to a first approximation the principal surfaces are planes normal to the axis. Let j be the common limiting value for ρ'/ρ and r'/r when the rays become paraxial. Then $\rho'/\rho = r'/r = j \sec\theta' \cos\theta$.

Also $\sigma = \mu'/\mu$ and $s = \frac{\mu'}{\mu} \sec\theta \cos\theta'$, so that for secondary planes,

$$\frac{y'}{y} = j\frac{\mu}{\mu'} . \sec\theta' \cos\theta,$$

and for primary planes

$$\frac{y'-Y'}{y} = j\frac{\mu}{\mu'} . \sec^3\theta' \cos^3\theta,$$

or, since θ and θ' are small,

$$\frac{y'}{y} - j\frac{\mu}{\mu'} = \tfrac{1}{2} j \frac{\mu}{\mu'} (\theta'^2 - \theta^2)$$

for secondary rays, and

$$\frac{y'-Y'}{y} - j\frac{\mu}{\mu'} = \frac{3}{2} j \frac{\mu}{\mu'} (\theta'^2 - \theta^2)$$

for primary rays.

The connection which has been shown to hold between the coma of a real image and the curvature of the surfaces of constant magnification close to the stop is only one example of numerous reciprocal relations which hold for all lens systems. Another one has been mentioned incidentally in the course of the proof, viz., the necessity of transferring to the stop the astigmatism it may be desired to remove from the image for which the system is to be corrected. Other examples which are more obvious are the relation between the coma in the stop and the distortion of the image, and the spherical aberration in the stop and the change of distortion as the magnification is varied. The particular instance which forms the subject of this note is dealt with at some length because a knowledge of the results here established is likely to be of use to optical computers at the present time.

ABSTRACT.

The absence of coma from an optical system free from spherical aberration can be established by means of the sine condition, from the refraction of rays which originate at the centre of the object. In actual systems spherical aberration is always present to some extent, and the conditions which have to be secured in practical cases do not involve mathematical freedom from spherical aberration and coma, but rather their confinement within predetermined limits. Previous investigations on this subject have been published by Conrady and by Chalmers; the present Paper shows that the conclusions of both require some qualification. Conrady's conclusion, that the sine ratio for any zone gives the exact magnification, holds only for rays in a secondary plane, and one of the further conditions (a) that the zone is free from spherical aberration, or (b) that the stop is at the principal focus for rays travelling in the negative direction, must also be satisfied. Under these conditions the magnification for rays in a primary plane is $\left(1+\tan\theta'\frac{d}{d\theta'}\right)$ times that for the secondary rays, where $2\theta'$ is the angle subtended by the zone at the centre of the image. The conditions assumed by Chalmers involve the satisfaction of a relation between various quantities, including the curvature and astigmatism of the system; his results may in consequence not apply strictly where very large apertures are involved. Analysis of the general problem shows that the primary and secondary comatic displacements can be accurately derived from the properties of thin oblique pencils of rays. If a central incident ray inclined at θ to the axis is met by an oblique ray in the same axial plane at a distance r from its point of origin, and is refracted at an angle θ' with the axis and meets the image plane at a distance r' from the image of its intersection with the oblique ray, the primary comatic displacement of the oblique ray plus the aberrationless first order displacement is proportional to $r' \sec\theta'/sr \sec\theta$, where $\mu s/\mu'$ is the magnification in a primary plane for a small normal object at the point of intersection of the two rays. The corresponding secondary displacement is $\rho'/\sigma\rho$, where the Greek letters have the same meaning for secondary rays as the corresponding Roman letters for primary rays. These two quantities determine the coma exactly whatever the spherical aberration may be. It follows that coma is dependent upon the principal surfaces of a lens. When the object is at infinity the coma is determined by the locus of the point of intersection of a ray, incident parallel to the axis, with its refracted portion.

The results established in the Paper form one of a series of reciprocal relations which exist between the aberrations of an object and those of the effective stop.

DISCUSSION.

Mr. S. D. CHALMERS pointed out the exact relation which he had given in the Paper quoted by Mr. Smith, and said that an assumption was necessary to apply this to the evaluation of coma. The assumption was that the two rays from an axial point and a neighbouring point to the secondary point on the stop would intersect in the image space substantially on the image of the stop. He thought this assumption was justified, though in

exceptional cases it might require careful examination. Apart from such special cases he thought it was perfectly justifiable to use the expansion for the coma in powers of sin A, and to deduce the values in the primary plane from these in the secondary plane. It would be necessary to determine the values for sufficient apertures in the secondary plane to find each of the terms in the coma expansion.

Prof. F. J. CHESHIRE said he was pleased to see that the subject of optical designing was occupying an important place in the work of the Physical Society. The maker who used to turn out the best microscope objectives in Germany would sit with his microscope beside him and a large number of component lenses from which he would try one combination after another until a satisfactory result was reached. In this way some excellent objectives and a large number of mediocrities were obtained. Abbe set out to kill this trial and error method, and introduced at the Zeiss works the system of determining mathematically the complete structural data beforehand. In this country a trial and error method was still largely employed. An approximate design was computed and a trial instrument was made. This turned out badly, and the designer had another try, and so on. First-class results were impossible on these lines. It was true that we had two or three first-class designers in the country whose designs could safely be made up without trial; and this must ultimately be aimed at by every optician if we are to take a prominent position in optical work.

Mr. BLAKESLEY said that an idea was involved in Mr. Cheshire's remarks which he had long held—viz., that a lens should only be used to form an image in the one particular position for which it is designed.

Mr. T. SMITH, in reply to Mr. Chalmers, said that the exact relation mentioned led at once to an expression equivalent to the alternative formula (17). If expressed in this form the equation was exact. The transformation given in Chalmers' Paper, which involved the neglect of a term, was apparently made to enable the primary coma to be found on the supposition that expansion in terms of sin A would separate out the coma of different ratios. Such a method could be applied provided the spherical aberration were also analysed into terms of different orders in sin A, but not otherwise. The process would then be a comparatively laborious method of reaching the result given immediately by either of the formulæ for the primary comatic displacement contained in the present Paper. The formula given by Chalmers was correct in the absence of spherical aberration; if the spherical aberration of the first n orders were negligible, no serious errors in the estimation of the first $(n+1)$ orders of coma from his formula would arise, but appreciable spherical aberration of the nth order would throw out the estimate of coma of the $(n+1)$th and subsequent orders.

XXIII.—*Chromatic Parallax and its Influence on Optical Measurements.*

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§ 4. Distinction between Chromatic and Positional Parallax.
§ 5. Constancy of TotalChromatic Parallax with Fixed Stop.
§ 6. Case of Cross Lines in a Dichromatic Field.
§ 7. Parallax in Optical Instruments.
§ 8. Methods of Avoiding Chromatic Parallax.
§ 9. Magnitude of Errors Involved.
§ 10. Possibility of Correcting Chromatic Aberration of the Eye.

§ 1. *Introductory.*

The purpose of this Paper is to discuss the origin of the well-known difficulties involved in making satisfactory measurements of the position of coloured lines or bands in the blue and violet regions of the spectrum. As an *ab initio* discussion of the phenomena, as they are met with in practical cases, would be complicated by the properties of the particular optical instruments in question, it will be desirable to discuss first of all some simple results, which, though of no practical importance in themselves, are nevertheless interesting, and involve nothing but the essential elements of the problem.

That the human eye is only very imperfectly corrected for chromatic aberration is well known; but the important part which this defect plays in many phenomena of vision is not so generally understood or suspected. In the following paragraphs some cases of parallax arising from this cause will be discussed, and their serious effect in diminishing the ease and accuracy of certain optical determinations will be demonstrated.

The chromatic properties of the eye are generally very similar to what they would be if its contents were replaced by water.* Usually the actual curves connecting focal length and wave-length are slightly flatter, from $\lambda=520\ \mu\mu$ to $\lambda=660\ \mu\mu$; but in some cases they are difficult to distinguish from the water

* P. G. Nutting, Proc. Royal Society, XC., p. 440.

curves. For a general treatment of the subject it will be fairly accurate to regard the eye as a single spherical surface with the refractive and dispersive properties of water. Thus, while the focus for red rays is slightly longer than for the yellow-green, the shortening in the blue and violet is very marked. In ordinary vision in white light the effect of this is not noticed, as the purple halo surrounding the image of each point is swamped by the light from adjacent points. It is when the eye is functioning under abnormal conditions of aperture, illumination, &c., as is usually the case in making optical measurements, that the phenomena to be described become noticeable.

§ 2. *Internal Chromatic Parallax.*

In Fig. 1 (a) let O be an object point simultaneously emitting (or reflecting) two monochromatic radiations—*e.g.*, one blue and one red—and let the optical system of the eye be represented by a single refracting surface of which B and R are the points conjugate to O for the two radiations.* In the diagrams the line of sight is made to coincide with the optic axis of the system, since the obliquity of 5 to 7 deg. which actually exists does not materially affect the phenomena.†

In general, the retina will lie somewhere between the blue and red images, B and R, its actual position depending on the remoteness of O and the amount of accommodation which is being brought into play. When the object is seen most distinctly, the position of the retina will be as shown in the diagram, where the blue and red diffusion patches are of equal size. So long as the pupil is filled with light there will be no relative displacement of the diffusion patches, however the eye may be moved ; for any motion of the eye can be represented by a simple rotation of Fig. 1 (a) about the point O, to which vision is direc ed.

In Figs. 1, (b) and (c), the effect of limiting the cone of light by a " pinhole " of 1 mm. to 2 mm. diameter, opposite either extremity of the pupil, is shown : the rays converge to B and R as before ; but the restricted pencils strike the retina in separate narrow patches at b and r. Thus, two separate " images," one red and one blue, are seen. Moreover, since the diffusion patches at r and b are of small diameter, the

* Of course, O is supposed to be at a distance from the surface large compared with the distances of B and R, which are, therefore, nearly coincident with the foci for these colours.

† See, however, reply to discussion.

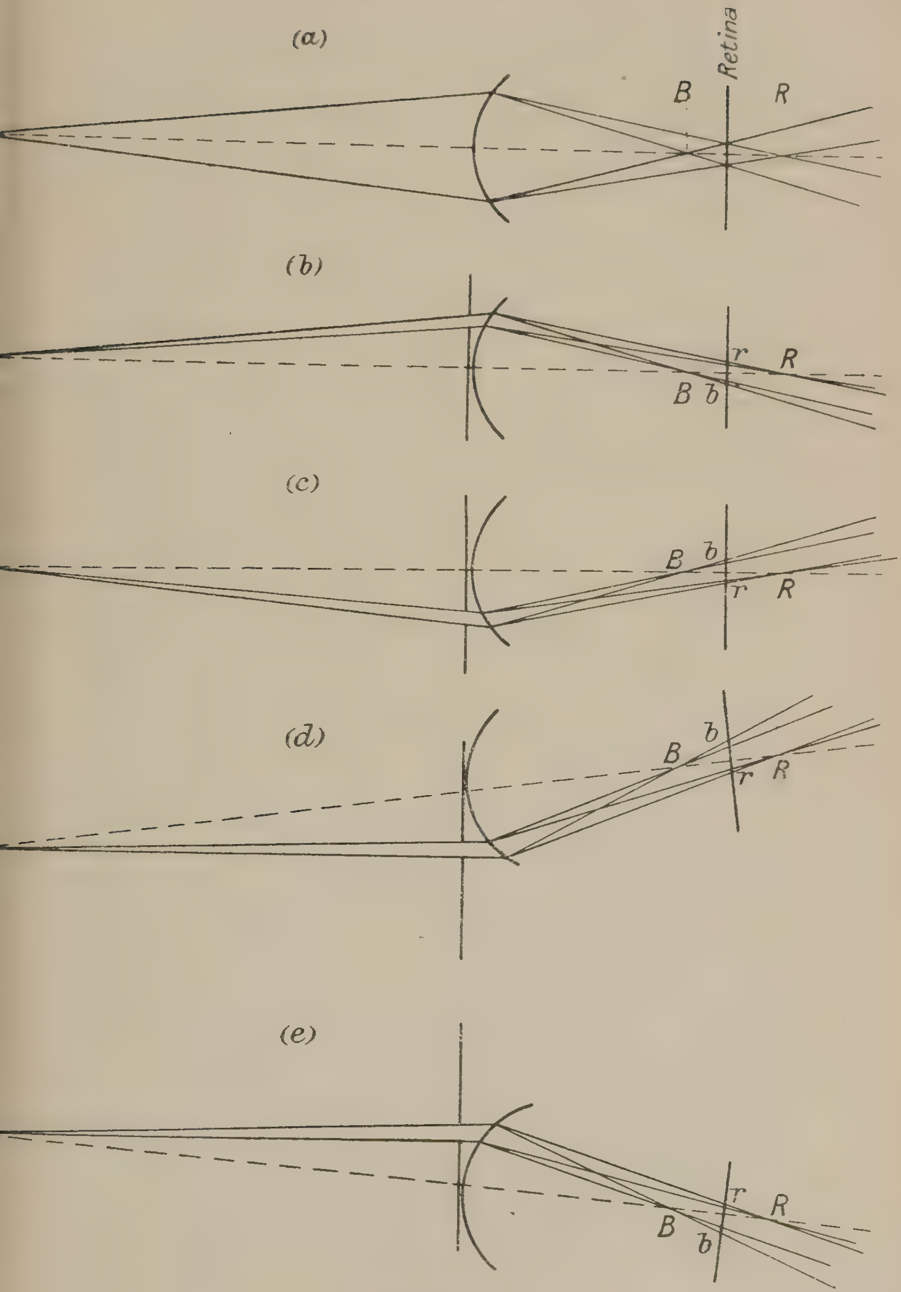

FIG. 1.

definition of each image is good. From the diagrams it is clear that the direction of the relative displacement is always such that the red image is on the same side of the axis as the pinhole. The mind interprets this as a separation of the object into two, the red one being always displaced in the *opposite* direction to the pinhole. Thus, if the pinhole be moved to and fro in front of the eye, parallax is observed between the red and blue "objects," which will only coincide when the pinhole is concentric with the line of sight.

Figs. 1, (*d*) and (*e*), show the effect of moving the eye behind a stationary pinhole. It is clear that these correspond exactly with the preceding two diagrams, except for a rotation of the whole diagrams about the object point. For the same *relative* motion of eye and pinhole, the same parallactic displacements will be observed whether it is the eye or the stop that is moved.

Hence, if an object seen in light of two colours, or two coincident objects of different colours, be viewed through a small stop of diameter considerably less than the pupil of the eye, parallax will be observed on moving either the eye or the stop. This type of parallax, since it depends entirely on the different paths pursued by the differently coloured beams within the eye, may be termed *internal chromatic parallax*. It may be exhibited in a variety of ways, of which the following are some of the more striking. If the filament of an ordinary electric lamp be viewed through a pinhole of about 1 mm. diameter, and two thicknesses of ordinary cobalt glass, a red and blue image of the filament will be seen.* These can be separated to a surprising extent by moving either the pinhole or the eye. To get the best results the experiment must not be performed in a room that is too bright; as in that case the pupil is small, and only small displacements of the stop to either side of the axis are possible. Also it is desirable to use two thicknesses of glass instead of one. With only one there is too much light from the intermediate regions of the spectrum, which detracts from the clearness of the red and blue images. Further, on account of the fact that the visibility of light diminishes, while the transmission of the cobalt glass increases, with diminishing wave-length, the "predominant wave-length" of the blue image is further in the violet with the double than with the single glass and the parallax is very much augmented. A single thickness of Wratten gelatine filter

* Cobalt glass has a transmission maximum in the red as well as in the blue and violet.

No. 35 does as well as the double cobalt glass; while this filter, in addition to one cobalt glass, is still more satisfactory.

Similarly, if the capillary of a hydrogen vacuum tube which gives a good G' line, or a slit illuminated by a mercury lamp, be looked at through a pinhole, parallax effects are observed between the blue lines and the other strong lines of the spectra. In fact, the eye working under these conditions acts as a spectroscope of which the angular dispersion depends on how far from the axis the small incident pencil passes.

Another illustrative experiment is to illuminate a slit with deep blue light (a monochromatic illuminator is very suitable) and examine it through a pinhole. On moving the latter, or the eye, the luminous strip leaves the slit (which is itself seen in the general illumination of the room) and oscillates to either side of it in a most striking manner. During these excursions the vacated slit appears to be quite black, as though no light were transmitted by it at all.*

§ 3. *External Chromatic Parallax.*

In the cases considered in the preceding paragraph, the two differently coloured incident pencils originated at the same object or at objects equidistant from the eye We shall now consider the case in which the objects are so arranged that both images are formed at the same distance within the eye, and may, in consequence, be simultaneously focussed on the retina, as in Fig. 2 (a). If both objects are in the line of sight the images r and b will coincide. Since rays from all parts of the pupil converge, for each object, to the same point on the retina, no parallax can be produced by limiting the vision by an eccentric stop, as shown in Fig. 2 (b). In this case, therefore, moving the pinhole across the eye causes no apparent displacement of the objects.

* In a recent Paper on "Visual Diffusivity" Phil. Mag., 33, pp. 18, January, 1917), H. E. Ives makes a casual reference to the danger of performing the experiments described in his Paper with a fixed stop in front of the eye on account of the apparent movements due to the chromatic aberration of the latter. In Prof. Gotch's "Purple-glass Test" (Burch's Physiological Optics, p. 48) the relative movements of the red and blue images of an electric filament, when observed through a pin hole and purple glass as described above, are employed to find the position of best focus for white light of an eye which has lost the power of accommodation. These are the only two cases of which the author is aware in which the effects of a relative motion of the eye and a small stop have been indicated, although the similar phenomena observed on eclipsing a portion of the pupil with the edge of a card or knife are described in many text books. The author is indebted to Mr. S. D. Chalmers for sending him the reference to Gotch's test after the reading of the Paper.

In Fig. 2 (*c*) the effect of a small displacement of the eye (with full pupil) is depicted The two images are still formed on the retina, but each must also lie on the line joining the corresponding object to the nodal point of the eye. A parallax,

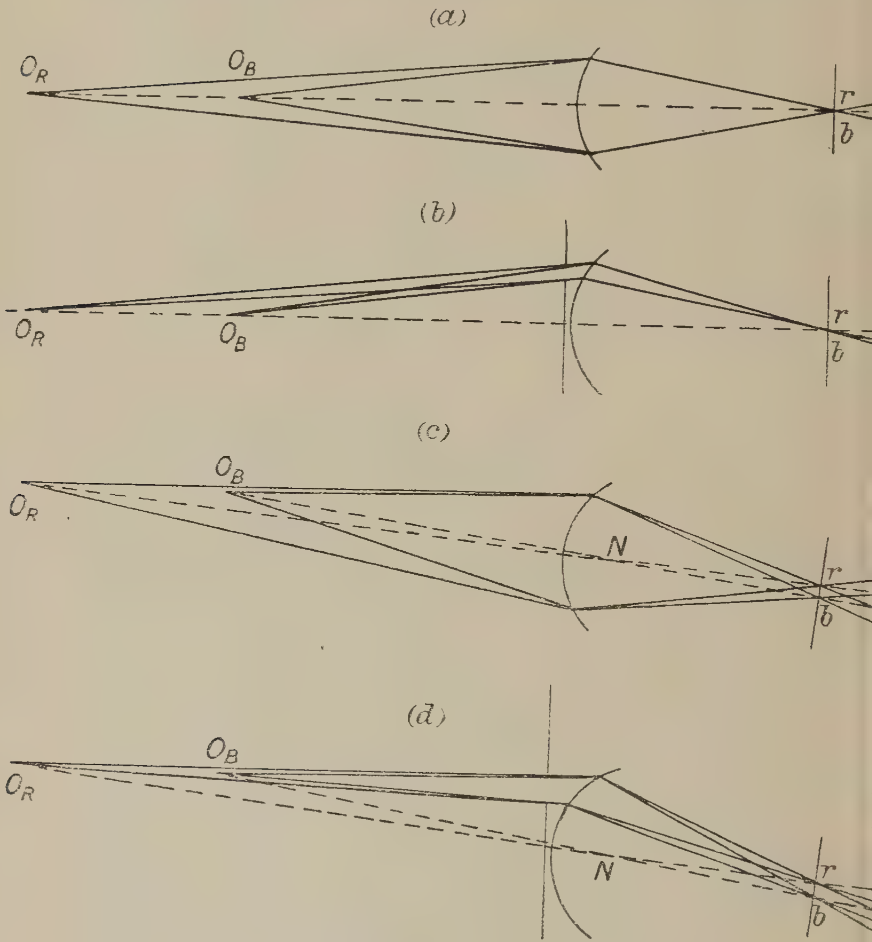

FIG. 2.

r b, is therefore the result. Here, again, since all the red rays intersect the retina at *r*, and all the blue rays at *b*, the interposition of a stop, as shown in Fig. 2 (*d*), will not affect the parallax; which, in this case, depends wholly on the angle

subtended by the objects at the nodal point, and not on what part of the pupil is being employed.

This type of parallax, observed between two objects at different distances from the eye, *but of which the images are both focussed on the retina at the same time*, may be termed external chromatic parallax to distinguish it from the type previously discussed.

§ 4. *Distinction between Chromatic and Positional Parallax.*

At first sight it may not be obvious that the parallax described in § 3, and termed external chromatic parallax, has anything to do with the colour properties of the eye, since it depends only on the angle subtended by the objects at the nodal point. The essential feature of this parallax is, however, that it is unaffected by the interposition of a small stop, and is, therefore, observed even when such a stop is situated on the line joining the objects, as shown in Fig. 2 (*d*). Further, it is not observed on moving the stop in front of a stationary eye. In the case of the parallax ordinarily observed between two non-coincident objects seen in light of the same wave-length (or same predominant wave-length), the effect of a stop is exactly the reverse of this. It is clear that if we have a small fixed stop placed in line with two similarly illuminated objects, no parallax can be observed on moving the eye behind it: because, since the incident narrow pencils are concentric, the refracted pencils, being equally refracted, must also be concentric, and will, therefore, intersect the retina in concentric diffusion patches, however far apart the objects may be. This case is represented in Fig. 3 (*a*). Fig. 3 (*b*) shows the effect of an eccentric stop on the apparent position of two objects situated in the line of sight and which would appear coincident if seen with the full pupil. There is an apparent displacement, due to the separation of the points i_1 and i_2 in which the refracted pencils, converging to the true images I_1 and I_2, meet the retina. The displacement depends on the distance of the stop from the axis, and so parallax is observed on moving the stop across the eye, even if the latter is stationary. Thus, in the ordinary case, parallax is due to a change in the relative direction of the mean rays of the incident pencil which reach the eye. In normal vision this is the effect of moving the eye; but if the fulfilment of this condition is prevented, as by the interposition of the fixed pinhole, moving the eye will not produce parallax; or, if the condition is ful-

filled without moving the eye, as by moving the pinhole, parallax will occur.

There is thus complete dissimilarity in the essential features of external chromatic parallax and that due simply to the difference in distance of two similarly illuminated objects from the eye.

The fundamental properties of the three types of parallax which may be encountered in suitable circumstances may be summarised as follows :—

Positional Parallax.—Object planes separate : image planes also separate : objects similarly illuminated : parallax produced if eye is moved with full pupil operative : or if a small stop is moved in front of stationary eye : not produced if eye is moved behind a fixed stop.

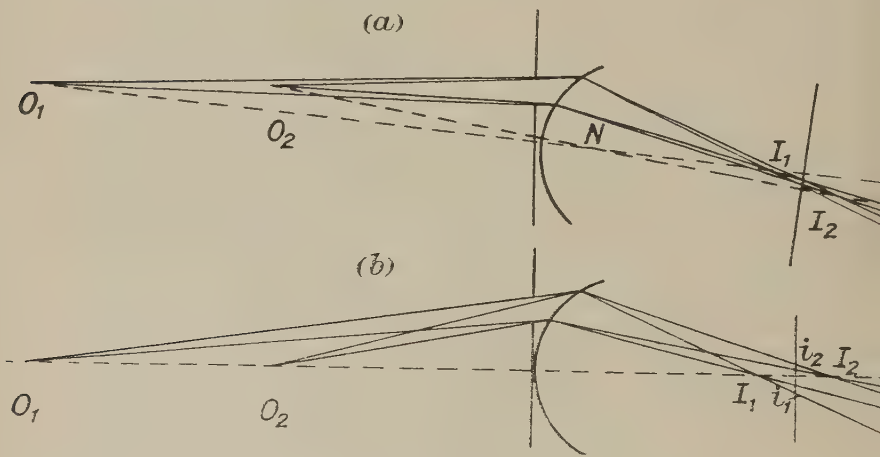

Fig. 3.

Internal Chromatic Parallax.—One object plane : two image planes : objects seen in light of different wave-lengths : parallax produced by any relative motion between the eye and a small stop. It cannot be observed with a full pupil.

External Chromatic Parallax.—Two object planes : one image plane : objects seen in light of different wave-lengths : parallax produced on moving eye either with or without a stop in front of it : not produced on moving the stop in front of the stationary eye.

§ 5. *Constancy of Total Chromatic Parallax with Fixed Stop.*

If neither the object planes nor image planes are coincident in the case of two differently coloured objects, the parallax observed on moving the eye behind a fixed stop will be neither internal or external in character, but will be compounded of both. As this is the most general case, it may be worth while investigating it quantitatively. The approximations which are made in the following treatment are self-evident :—

Let R and B, Fig. 4, be two objects (red and blue respectively) situated near the line of sight. Let R' and B' be the points in which the axis is intersected by RD and BD, the princ pal rays of the thin pencils which enter the eye through the small stop at D. The principal rays of the refracted pencils must cross the axis at R'' and B'', the points conjugate to R' and B'. The points r and b, where these rays meet the retina, are the centres of the observed images ; and the angle

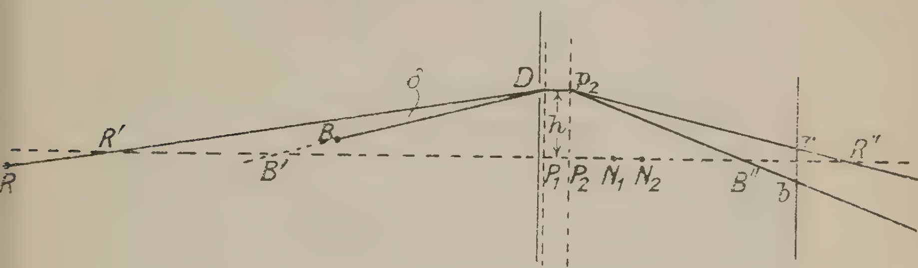

FIG. 4.

which rb subtends at the posterior nodal point N_2 is the apparent angular separation of the objects.

In the figure, the blue object is nearer the eye than the red one ; but is still so remote that the blue image is anterior to the red image. This is, therefore, a case intermediate between internal and external chromatic parallax. The analysis holds, however, for any order of objects and images.

P_1, P_2, N_1 and N_2 are the principal and nodal points of the system. Denote P_1R' by U_r ; P_1B' by U_b ; P_2R'' by V_r ; and P_2B'' by V_b. Let μ_r and μ_b be the refractive indices of the final medium for the two wave-lengths, and μ its index for the mean rays of the spectrum (say sodium light), and let f_r, f_b and f be the corresponding values of the first principal focal length.

Then, from the general formula $\frac{1}{U}+\frac{\mu}{V}=1/f$, we obtain

$$\frac{1}{V_r}=\frac{1}{\mu_r}\left\{\frac{1}{f_r}-\frac{1}{U_r}\right\} \text{ and } \frac{1}{V_b}=\frac{1}{\mu_b}\left\{\frac{1}{f_b}-\frac{1}{U_b}\right\}.$$

The angle $rN_2b \doteqdot \mu \times$ angle $rP_2b \doteqdot \mu \times$ angle rp_2b,

$$=\mu\left\{\frac{h}{P_2B''}-\frac{h}{P_2R''}\right\}=\mu h\left\{\frac{1}{V_b}-\frac{1}{V_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b}\left(\frac{1}{f_b}-\frac{1}{U_b}\right)-\frac{1}{\mu_r}\left(\frac{1}{f_r}-\frac{1}{U_r}\right)\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}+\frac{\mu_r U_b-\mu_r U_r}{\mu_b\mu_r U_b U_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right\}-\frac{\mu}{\mu_b}\left\{\frac{h}{U_b}-\frac{h}{U_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_1 f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right\}-\frac{\mu}{\mu_b}\delta,$$

where δ is the angle subtended by the objects at the centre of the small stop.

The apparent angular separation of the objects is, therefore, made up of two parts. One of these, $\frac{\mu}{\mu_b}\delta$, is constant for any one arrangement of the objects, and is simply the angle which they subtend at the stop, since the fraction $\frac{\mu}{\mu_b}=0{\cdot}995$ approximately, and may be taken as unity. The other part, depending on the sign and magnitude of h, is the parallax obtained by moving the eye behind the stop, a distance h off the axis. It would clearly vanish, for all values of h, if the system were achromatic, or if both wave-lengths were the same; and is, therefore, wholly chromatic in origin; bearing out the result arrived at in § 4 that a fixed stop prevents the observation of purely positional parallax, so long as all the light passing through the stop enters the pupil.

Since $\qquad \frac{1}{f_r}=\frac{\mu_r-1}{\mu-1}\cdot\frac{1}{f}$; and $\frac{1}{f_b}=\frac{\mu_b-1}{\mu-1}\cdot\frac{1}{f}$,

the expression for the parallax may be written

$$\mu h\left\{\frac{1}{(\mu-1)f}\left(\frac{\mu_b-1}{\mu_b}-\frac{\mu_r-1}{\mu_r}\right)+\frac{1}{U_r}\left(\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right)\right\},$$

which reduces to

$$\mu h\left(\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right)\left(\frac{1}{(\mu-1)f}+\frac{1}{U_r}\right).$$

To obtain an approximate numerical estimate of this quantity we may assume the refractive properties of the living eye to be those of distilled water about 40°C.

These are

$$\mu_C=1{\cdot}32878\,;\ \mu_D=1{\cdot}33061\,;\ \mu_F=1{\cdot}33473\,;\ \mu_{G'}=1{\cdot}33793.$$

If we take C and G' as the red and blue wave-lengths respectively, we get for the parallax*—

$$\pi=h\times1{\cdot}33\left(\frac{0{\cdot}00915}{1{\cdot}3288\times1{\cdot}3379}\right)\left(\frac{1}{0{\cdot}3306}\cdot\frac{1}{17}+\frac{1}{U_r}\right)$$

$$=h\times0{\cdot}00685\left(0{\cdot}178+\frac{1}{U_r}\right).$$

The coefficient of h is constant except for the effect of variations of U_r. If this varies from ∞ to 250 mm.—the near point for normal eyes—the factor in brackets only ranges from 0·178 to 0·182, a variation of a little over 2 per cent. Thus, to within a small percentage, the parallax between two differently coloured objects, seen through a small fixed stop, is independent of the position of the objects.

Its value may be taken as—

$$\pi=h\times0{\cdot}00685\times0{\cdot}178=h\times0{\cdot}00122 \text{ radians}=h\times4{\cdot}2 \text{ minutes}.$$

With a total motion of the pupil of 5 mm. to 6 mm., a parallactic range of 20 to 25 minutes should be encountered. The full range which the author has observed in various optical instruments lies between 15 and 27 minutes, when corrected for magnification; depending on the size of the pupil at the time. This indicates how closely the assumptions made as to refractive properties fit the actual case.

§ 6. *Case of Crosslines in a Dichromatic Field.*

The results of the preceding paragraphs concerning coloured objects are clearly applicable to the case of dark objects seen against coloured backgrounds. An important case of this type is that of the crosslines of optical instruments.

To study the properties of crosslines under various conditions of illumination the arrangement of Fig. 5 is very convenient. A transparent grating, G, is mounted between two lenses

* The mean focal length of the eye is here assumed to be 17 mm., the figure given in Tscherning's Optique Physiologique. In the schematic eye the focal length is 15·5 mm.; while in the "reduced" eye of Donders it is only 15 mm. Either value would give a greater figure for the chromatic parallax than the one adopted.

L_1 and L_2 to act as collimator and telescope objectives respectively. Two gas-filled half-watt tungsten lamps, with short straight lengths of spiral filament, are mounted on stands so that the first order spectra on opposite sides of the normal, due to the two sources, are focussed at P, the observing pinhole An eye placed behind P sees the back of the lens uniformly illuminated by a mixture o the two wave-lengths which are focus ed at P. By adjusting S_1 and S_2 any two wave-lengths can be combined. A rheostat in each lamp circuit enables the relative intensity to be varied.

The crosslines W of thin copper wire are mounted on a diaphragm, and placed in front of the lens L_2.*

If we commence with a monochromatic blue field, and add a gradually increasing proportion of light of some longer wave-length, say green or red, we find when the proportion of this

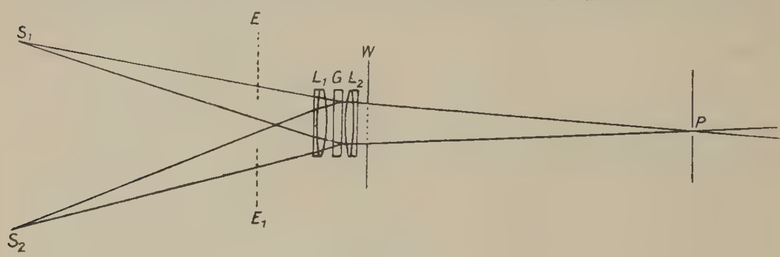

Fig. 5.

colour reaches a certain value that two sets of wires are visible, which exhibit parallax on moving the eye behind the stop. As the second colour increases in intensity the two sets become equally well marked. Finally, when the longer wave-length predominates strongly, only the second mage can be seen, the original one being blotted out. If the added illumination is white light, obtained by placing one of the lamps, so that its direct image is formed at P, the result is similar to that described; since white light, for focal purposes, and consequently for parallax effects, is equivalent to a monochromatic illumination of wave-length somewhere in the yellow green, depending on the nature of the source. In general, these effects will be produced if the blue field is diluted with any light, monochromatic or mixed, of different predominant wave-length.

By interposing vertical screens, shown dotted at E and E',

* The crosslines should be of the diagonal type; not vertical and horizontal.

each colour may be confined to half of the field; and the behaviour of cross lines, set on the edge of a coloured band, when the rest of the field is illuminated with light of different wave-length, may be observed. This consists of a breach of continuity of the lines at the junction of the colours, the parts of the cross on each side of the boundary separating or overlapping as the edge is moved. The relative motion of the fields themselves as well as the cross wires is very striking in the experiments with this apparatus. Further, when working with blue or violet fields, a third image of the aperture is seen due to the large proportion of undiffracted white light which is scattered by the rulings of the grating. This in nowise affects the relative movements of the *coloured* images, which behave quite independently. If both of the spectrum colours are blue, the scattered white light can be cut out by a cobalt glass filter if desired. With the arrangement employing the screens E and E', the difference in wave-length required at any part of the spectrum, before appreciable parallax effects are introduced was determined. One half of the field was illuminated with light of a certain wave-length, and, by moving the appropriate lamp, the wave-length of the other half was adjusted until no discontinuity of the cross-lines could be produced by moving the eye. The results showed that at $\lambda=410\mu\mu$ the difference of wave-length, in the direction of increasing λ, at which parallax could just be detected was only $5\mu\mu$; at $\lambda=430$ this had increased to $10\mu\mu$; at $\lambda=450$, to $15\mu\mu$; at $\lambda=500$, to $20\mu\mu$. It is difficult to detect parallax between two colours of wave-lengths greater than $560\mu\mu$ until one of them reaches the extreme red. The above results apply, of course, only to the author's eyes, but they are probably typical of most normal cases.

§ 7. *Parallax in Optical Instruments.*

A large proportion of the work in practical optics involves the determination with the highest possible precision of the position of spectrum lines, or the edges of coloured bands, in the field of view of a telescope. In properly designed instruments, in order to make full use of the available resolving power, the magnification of the telescope employed is usually from 20 to 25 linear, for each inch of effective aperture of the dispersive system. A real image of the face of the last prism, or whatever limits the effective aperture of the beam entering the object glass, is formed by the remainder of the optical system, and can

be seen just in front of the eyepiece, if the eye is held about 9 in. away. All the light passing through the system must pass within the limits of this image, which is termed the "exit pupil" of the instrument; and which acts as a fixed stop through which vision of the field of view takes place. For the whole of the field to be seen, the pupil of the eye must be coincident with, or at least somewhere near, the exit pupil of the telescope. The diameter of the exit pupil in any direction is equal to the diameter of the effective aperture in the same direction divided by the magnification of the telescope. Its horizontal diameter is never, therefore, much more than 1 mm. in instruments of precision It follows from this that ordinary parallax, due to want of simultaneous focus of the cross lines and objects in the field of view, can never be observed in such instruments. The following experiment illustrates this: set the cross lines of a spectroscope, such as a Hilger constant deviation instrument, on a bright yellow or green line, using a fairly wide slit, so that there is light enough from the line itself to see the cross lines by. It will then be found, however far out of focus the line may be, that there is no parallax observed on moving the eye, except for spasmodic jerks at the very extreme range of the motion, due to the fact that at these points the edge of the iris is eclipsing part of the exit pupil, and thereby altering the relative direction of the principal rays of the effective pencils; which, we have seen, is the essential condition for positional parallax: but so long as the position of the eye permits all the light transmitted by the exit pupil to enter it, there is no parallax.

If, however, the cross wires are illuminated with light of different refrangibility from the line or band under observation, chromatic parallax effects will be observed; and these, as shown in § 5, cannot be eliminated by varying the focus either of cross wires or spectrum lines. Unfortunately, this is almost always the case, from a variety of causes, when measurements are being made in the blue and violet part of the spectrum. In the case of instruments such as the ordinary spectrometer, when spectrum lines are reduced to the fineness essential for accurate work, the cross hairs are usually quite invisible; and it is necessary, before settings can be made, to supply some general illumination in the field of view. This, as far as the author's experience goes, is almost invariably accomplished by arranging a table lamp so that it throws sufficient light into the telescope to render the cross hairs distinctly visible; the result

being that large parallax effects are produced with blue and violet lines. In some cases where there are many intense lines in the most luminous part of the spectrum, there may be sufficient illumination, due to the light scattered at the surface and within the material of the lenses and prisms, and also to light of other colours reflected from the sides of the tubes, to make the cross wires visible in the blue and violet regions, without the addition of extraneous light. The result, however, is the same, since the scattered light is undispersed and of quite different predominant wave-length from the lines under examination.

The serious hindrance which this puts in the way of accurate observations on lines of short wave-length is one with which most optical experimenters are familiar. The obvious result is that much greater variation of individual readings is encountered than when working in the yellow or green; and correspondingly longer series of observations must be made. There is, however, a more serious source of uncertainty, which patience in taking a multiplicity of readings will not eliminate. An observer, when using only a limited portion of his pupil, has a tendency to place his eye in one position more than in others; the favoured position being that in which the small pencil of light encounters fewest optical imperfections in its way to the retina, and produces, in consequence, the best image. The factors which determine the path of best definition will be the existence of any of the imperfections grouped under irregular astigmatism; and also the temporary arrangement of "muscæ volatantes," striæ in the humours, &c. Such differences will be slight, and individual placings of the eye will vary considerably; but the mean to which the observer's settings will always approximate, if sufficient readings be taken, will be that corresponding to the favoured position of the eye. In general, this favoured setting will not be that in which the principal rays pass through the nodal points of the eye, and there will, therefore, be a difference in the ultimate result which the observer obtains under these conditions from that which he would obtain were chromatic parallax absent. Moreover, this error varies from time to time, probably on account of variation in the distribution of the temporary irregularities mentioned above, and possibly others. It is therefore impossible to determine a reliable correction to be applied to readings made by a particular observer under specified conditions.

Before proceeding to deal with the simple methods which can be employed to obviate all difficulties due to chromatic parallax, reference may be made to the erroneous method which students of practical physics sometimes attempt to employ for the focussing of telescopes—viz., to adjust until there is no parallax between the line, or other object in the field, and the cross wires. With very low-power telescopes, such as are sometimes found on instruments intended for the use of students, the process may meet with a limited amount of success; but the method is essentially wrong, and cannot be applied at all with telescopes of which the magnification bears the most suitable ratio to the aperture. In such cases the only criterion of exact focal adjustment is simultaneous sharpness of definition of object and crosshairs, and a critical judgment of definition is one of the most generally useful faculties that a student of optics can acquire.

§ 8. *Methods of Avoiding Chromatic Parallax.*

The essence of such methods clearly lies in supplying illumination for the cross wires of the same wave-length as the line on which they are to be set; and in excluding from the field all stray light of other wave-lengths if it is present to an appreciable amount. The most suitable method of fulfilling these conditions depends on the circumstances. If the wave-length concerned is not very far in the blue it will sometimes be sufficient to throw some white light from a table lamp into the field and employ a filter of cobalt glass in front of the eye-piece. This may bring the predominant wave-length of the field illumination sufficiently near to that of the line to destroy most of the chromatic parallax between it and the cross wires. At the red end of the spectrum a piece of red glass or, preferably, a Wratten gelatine filter, No. 25, mounted between glass, is all that is ever required, since the effects at this end are very much less marked than with short wave-lengths. For most of the blue and violet, however, the small difference in wave-length necessary to produce parallax, renders the cobalt glass practically useless. A method of general applicability for illuminating the cross hairs in instruments of the spectrometer type, is to arrange an auxiliary source of white light, so as to produce a continuous spectrum as a background to the line spectrum. By this means the cross wires are always illuminated by light of the same wave-length as the line on

which they are set, and no parallax effects are present. The observations are then as satisfactory, other things being equal, as those made at other parts of the spectrum. A suitable arrangement is shown in Fig. 6. The line source S_1 is focussed on the slit by the lens L_1 in the usual way. The white light source S_2 is focussed on the slit via the 45 deg. reflector R, consisting of a piece of unsilvered microscope cover glass. A very intense source at S_2 is necessary in order, when using a fine slit, to get a sufficiently bright background as far in the violet as is sometimes required. If the lines are bright enough to stand some cutting down, the continuous spectrum can be greatly intensified by semi-silvering the reflector ; but it is better in this case to use for R a piece of

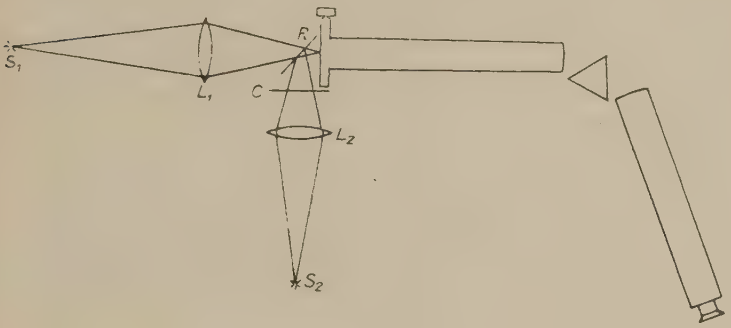

FIG. 6.

thin mirror glass, from which the silver has been scraped off, in parallel strips of 1 mm. to 2 mm. broad. If the width of the silvered and unsilvered strips are equal, the reflected and transmitted intensities are nearly equal. Reflectors can be made giving any desired proportion to the intensities. With this type of reflector a satisfactory background can be obtained, well beyond the H and Hg lines at 434 and 436 $\mu\mu$, from a "Pointolite" tungsten lamp. To get much beyond this—to include, for instance, Hg 404—or, in cases where the lines are so faint that only the plain glass reflector can be employed, it is essential to use a strong carbon arc. If a strip reflector is used, it must be further from the slit than shown in the diagram, so that the strips are not visible in the field of view. The source S_2 must be shaded by a sheet iron or asbestos cover, which only lets light out in the required direction ;

otherwise, the general blaze which it creates is very annoying. It is further desirable to insert a cobalt glass screen at C in order to cut out the most luminous constituents of the light scattered within the instrument. The illumination of the field due to scattered light is, of course, much the same at whatever wave-length the instrument is set, but it is quite swamped in the luminous parts of the spectrum by the relative brilliance of the spectrum colours. In the blue and violet, however, the luminosity of the spectrum colour is so low that the scattered light is relatively intense; and, in most instruments, illuminates the cross lines quite distinctly, even beyond the limits of the visible spectrum. If this stray white light is not absorbed by the filter C, the effectiveness of the spectrum illumination is impaired in the blue, and nullified entirely in the violet. A modification of the above arrangement is to employ a separate collimator for the white light spectrum, placing it so that the parallel beam emerging from it can be directed into the prism system by the glass reflector. This has the advantage that a wide slit can be employed for the continuous spectrum, and the necessary intensity obtained with a very much less powerful source of light. Owing to the adjustments required, it is a troublesome method, except for a permanently set up apparatus. It has the further disadvantage that the reflector must be optically flat on both sides, or the definition of the line spectrum, the light of which traverses it between the collimator and prism system, is spoilt.

While the above methods are probably the most satisfactory that can be employed where circumstances make it essential to employ fine lines, there is, nevertheless, a very much superior method of using a spectrometer, which is applicable to almost all purposes for which such instruments are now used. This consists in replacing the stereotyped fine slit by a wide slit, ·5 mm. to 1 mm. in width, with a central spider line mounted parallel to the jaws. Each spectrum line is represented in the field of the instrument by a broad coloured band with a fine black line down the middle. The cross lines are set on this line. This method has many advantages over the use of fine bright lines even under the most ideal conditions. In the first place it entirely prevents the occurrence of chromatic parallax without the necessity of employing auxiliary illumination for the cross lines; since the latter are seen sharply defined on the bright band of spectrum colour. Further,

the comfort experienced in setting black cross hairs* on a black line under these conditions of illumination, is of the very greatest importance in maintaining the observer's efficiency during prolonged series of observations Everyone is aware of how, in setting on bright lines even under the best conditions as to illumination of cross lines, there is always a feeling of uncertainty when the setting is actually accomplished. Just before hand, when the cross hairs and line are distinctly separate, both appear ideally sharp, and the observer feels that superposition can be effected with the greatest accuracy; but when the superposition has actually taken place, diffraction effects make themselves evident in a wavering and loss of definition at the point of intersection, which produces an annoying lack of confidence in the accuracy of the setting, and is, in addition, very trying to the eyes. With the arrangement described there is no such effects at the moment of contact: everything appears as distinct and clear as if it were a diagram drawn on paper. Further, the tiresome fluctuations of accommodation, to which the eye is subject when using faintly illuminated cross lines, are practically absent.

The method has also the advantage that the full brightness of any line is utilised. With a slit cut down so as to give a fine line spectrum the intrinsic brightness of the lines is very much reduced. Since it is essential to keep the illumination of the cross wires faint enough in comparison with the line to preserve a satisfactory contrast, it is clear that with lines of low visibility the wide slit method gives a much more satisfactory working illumination, since, in that case the full brightness of the line is utilised in lighting the cross wires, instead of something which must be considerably less bright than the line, even with the latter at reduced intensity.

This method can be employed with advantage for all such purposes as the measurement of refractive index and dispersion on the spectrometer. The full separating power of the instrument is obtained, since the spider line is of negligible width. With sodium light, for instance, the broad yellow band of superposed D_1 and D_2 is traversed by the black lines due to the two components, each as clearly defined as if the other

* The author refers throughout to cross lines mounted diagonally. Unfortunately, there are still many instruments made in which the cross lines are vertical and horizontal. It is impossible to make accurate settings with such lines unless a slit of appreciable width be used and its image bisected by the vertical cross line. See, however, reply to discussion in connection with a setting of this type.

component were not there. Except with spectra of great complexity, such as that of the iron arc, the superposition of the bands of neighbouring lines does not give any trouble.

The same type of setting is suitable for use in the measurements of the angles of prisms ; and here, as in the case of refracted lines, it gives a marked gain in comfort and accuracy compared with the stock method of using the image of a fine slit.

The spider line may conveniently be mounted permanently between thin glass plates, or the fine cross lines ruled on glass which replace spider lines in some instruments may be used. The author has sometimes employed for this purpose the finely divided micrometer scale—1 cm. divided to millimetres and tenths of millimetres—from a Zeiss microscope eyepiece. The scale is attached across the slit with the rulings parallel to the jaws. Each spectrum band is now crossed by an image of part of the scale, and a suitable graduation is chosen on which to make settings. On account of the extreme fineness of the rulings this proves very satisfactory.

Spider lines, while giving excellent results with instruments in which the definition is very good, must be replaced by something coarser when the lenses or prisms are not of the best quality, and in low-precision instruments generally. The author has found that two or three quartz fibres, of thickness varying from 5 to 20 thousandths of a millimetre, are suitable for most purposes. Even for the finest work there does not appear to be any great advantage in using a line finer than 7 or 8 μ ; this, in fact, being a very convenient thickness to employ in work demanding the greatest precision. The plates between which the fibre is mounted should be separated by at least a millimetre to avoid interference effects, and the one next the slit should either be of optical glass or of thin microscope cover glass ; otherwise the sharpest definition of the line will not be obtained. Two transverse fibres—or lines ruled on the nearest glass—should also be supplied. They should be about 0·5 to 1 mm. apart, and serve to indicate the middle of the slit. Adjustments should be made so that the cross lines in the eyepiece bisect the vertical interval between these transverse lines.

An instrument of great practical importance in which chromatic parallax is often extremely troublesome is the Pulfrich refractometer, which is at present employed in the commercial

testing of optical glass. In the field of view of this instrument each spectrum colour gives rise to a more or less broad band, on one edge of which the cross lines have to be set. There is usually a very large proportion of stray light in the field, due to scattering of the incident beam by the matt surface surrounding the polished circle on which the specimen is placed. This illuminated surface can usually be seen sending light straight into the telescope if the eyepiece is removed. The defect is worst at the larger angles, owing to the diminution of the effective aperture of the polished surface with increasing obliquity. Some of this scattered light can be cut out by a judicious use of the revolving cap; but it is usually impossible, except within a limited range of angle, to avoid the presence of sufficient, from this cause alone, to produce serious parallax troubles with blue and violet lines.

With the hydrogen spectrum, as usually obtained from commercial tubes, the G' line is so faint that it is rarely possible to see more than one image of the cross lines, which moves about with respect to the edge of the band as the eye is moved. With a brighter band, obtained from a heavily run tube under proper conditions of pressure,* the various effects described in § 6 are usually obtained.

Unfortunately the nature of the optical system of the Pulfrich refractometer precludes the possibility of using the improved type of setting, which has been described for instruments of the spectrometer type. The best that can be done in this case is to interpose a piece of cobalt glass in front of the specimen to cut out the most luminous constituents of the stray light. It is not usually necessary to supply continuous spectrum illumination when using hydrogen, as the overlapping bands of the secondary spectrum supply sufficient continuous background to render the cross lines visible. The reduction in intensity of the G' band itself, due to the absorption of the cobalt glass, makes it essential to use very heavy discharges; but when this is done settings practically free from the disturbing effects of parallax, and usually as accurate as those on any other line, can be made. If it is necessary, in any case, to supply spectrum illumination for the cross lines, this can be effected by replacing the right angle prism for use with flames by a thin glass plate, and arranging a half-watt tungsten lamp so as to produce a continuous spectrum in the

* *See* Proc. Phys. Soc., Vol. XXVIII., p. 60; or Collected Researches National Physical Laboratory, Vol. XIII., p. 53.

field of the refractometer. The brightness of this can be controlled by means of a rheostat in the lamp circuit, to give a suitable background for the cross lines at any part of the spectrum. The lamp should be shaded in all but the one direction to prevent glare.

§ 9. *Magnitude of Errors Involved.*

Many sets of observations have been made by the author, and by several of his colleagues, to determine the errors liable to arise in using instruments of different types, if chromatic parallax is not eliminated. A brief summary of the results will suffice to show the seriousness of these. In § 5 it was shown that a total range of parallax of over 20 minutes in apparent angle might arise in certain circumstances from this cause. With an optical instrument this figure has to be divided by the magnification to obtain the equivalent range in the actual settings. For example, in a particular spectroscope, of the constant deviation type, furnished with a micrometer eyepiece, the extreme range of the settings that could be made on the G' line, with a reddish white background to the cross lines, was, with a dark adapted eye, 0·14 mm. on the micrometer or 1·8 minutes in angle. The power of the telescope was 15, so that the apparent angular range was 27 minutes. With white illumination of the field from a table lamp, about three-quarters of this range was obtained. In fact, the effect does not diminish greatly until the predominant wave-length of the field illumination becomes fairly near that of the line. When chromatic parallax was absent settings could be repeated to 0·001 mm. The maximum difference obtainable from the true readings with improper illumination was therefore, in this case, about 70 times the observational error under proper conditions. Of course, in actual determinations the eye would never be used near either extreme of the pupil. As an example of the discrepancies which may be encountered in practice, the following observations were made on the mercury line at λ, 436$\mu\mu$. No adjustments were altered during the whole set of observations; which are, therefore, comparable one with another. The readings are the arbitrary divisions on the micrometer head, 0·1 corresponding to about 1 second. In series A, the field was illuminated with white light from a table lamp ; in B the same lamp was used, but the light from it was passed through a Wratten No. 49 filter, which is better for this line than cobalt glass ; and in C spectrum illumination

was used. In each case the greatest care was taken to get as satisfactory settings as possible, and the cross wires were always moved aside, before applying the eye to the eyepiece, to prevent one setting from biassing the next. The table shows that in the first two series, in which chromatic parallax

Series A. White light in field.		Series B. Filtered light in field.		Series C Spectrum illum. of field.	
Observations.	Numerical difference from mean.	Observations.	Numerical difference from mean.	Observations.	Numerical difference from mean.
17·4	0·2	18·4	0·7	17·4	0·1
16·2	1·2	17·7	0·0	17·4	0·1
17·5	0·1	16·8	0·9	17·2	0·1
17·1	0·5	17·4	0·3	17·3	0·0
15·9	1·7	17·5	0·2	17·3	0·0
18·0	0·4	17·8	0·1	17·2	0·1
19·1	1·5	17·7	0·0	...	...
18·4	0·8	16·9	0·8	...	...
18·	0·4	18·1	0·4	...	...
16·9	0·7	17·7	0·0	...	...
17·5	0·1	18·0	0·3	...	...
17·6	0·0	18·2	0·5	...	...
15·8	1·8	18·6	0·9	...	...
17·1	0·5	18·7	1·0	...	...
19·4	1·8	17·3	0·4	...	...
18·6	1·0	17·3	0·4	...	...
16·8	0·8	17·7	0·0	...	...
18·5	0·9	18·1	0·4	...	...
17·5	0·1	17·8	0·1	...	...
18·0	0·4	17·0	0·7	...	...
Means 17·6	0·7	17·7	0·4	17·3	0·1

was present, the probable errors of individual settings were respectively seven and four times that of series C, in which it was not present. Further, the difference between the mean values of series A and B from that of series C—which may be taken as the true reading—shows the danger of relying on the mean, even of an extended series of observations.

The results show that the variations in individual settings are much reduced by the use of the filter, but that they are still several times in excess of what they should be.

Results of a similar character have been obtained for various types of instrument by the author's colleagues. For example, readings for the G' line, obtained on the Pulfrich refractometer by certain observers, sometimes showed persistent differences of eight or nine-tenths of a minute—a discrepancy at least five times as great as is tolerable in these determina-

tions. Usually the agreement was much better, but it is only when the proper conditions of illumination are secured that results can be obtained of an accuracy comparable with that for other lines.

§ 10 *Possibility of Correcting the Chromatic Aberration of the Eye.*

By employing an eyepiece of suitable design, or by the provision of a suitable spectacle lens, the chromatic aberration of the eye could be corrected in the sense that differently coloured objects in the same object plane would have their images simultaneously focussed on the retina. It is not at present known, on account of insufficient data, to what extent vision would be improved by the employment of an achromatising lens in front of the eye; or whether the definition of optical instruments, such as telescopes and microscopes, could be improved by introducing the chromatic properties of the eye into the computations; but it is highly probable, on general grounds, that an appreciable improvement would result in many cases. At any rate, it is desirable that accurate information of the chromatic properties of the eye, and the range of variation met with in different persons, should be available. The author had commenced an investigation of methods suitable for such determinations over a year ago; but the work is at present in abeyance for lack of time.

But whatever advantages might or might not accrue from some method of compensating the chromatic aberration of the eye, it is, unfortunately, the case that no such method is likely to obviate the difficulties from this cause which have been discussed in the preceding paragraphs. Let the eye-piece be so designed that coincident objects of different colours in the field of view are finally focussed on the retina. This is only accomplished by altering the positions of the virtual objects with respect to the eye; and this, as we have seen, has no effect on the parallax. If, on the other hand, we were to achromatise the eye by means of a spectacle lens, fixed to the head by the ordinary framework, and partaking of all the movements of the eye, the essential condition for absence of parallax—viz., that the moving system shall be achromatic in itself—would be fulfilled; but the achromatising lens would require to be exactly coaxial with the eye, otherwise parallax errors would result; for it is possible to arrive at the condition in which the lens and eye are not coaxial, by starting from the condition in

which they are coaxial and moving the eye while the lens remains fixed. During this motion the moving system being no longer achromatic in itself, a relative displacement of the two coloured images takes place. Thus, the apparent separation of the objects, when the lens is not coaxial with the eye, differs from its true value, and settings will be in error. As it is not practically possible to ensure exact centering, with respect to the axis of the eye, of anything in the nature of a spectacle lens, the cure of chromatic parallax cannot be effected optically, and recourse must be had to devices such as have been described in § 8.

The author wishes to record his indebtedness to his colleagues Mr. T. Smith, Dr. J. S. Anderson, and his late colleague Mr. R. W. Cheshire for their interest in the problems with which the Paper deals ; and to Dr. Anderson, Mr. R. J. Trump, Mr. D. W. H. Bell and Miss A. B. Dale for their patience in taking long series of test observations on which some of the conclusions have been based.

ABSTRACT.

The Paper describes various parallax effects observable when objects illuminated by light of different colours are seen through a "pinhole" of smaller diameter than the pupil of the eye. The effects are due to the well-known chromatic aberration of the eye ; and it is shown that in most optical instruments of precision the necessary conditions for parallax of this nature are present when observations are made in the blue and violet ; and that this results in a serious diminution of the accuracy of such measurements. Several methods of obviating the difficulty are described.

DISCUSSION.

Prof. CHESHIRE said it was possible that the author had laid his finger on one of the main causes of the discrepancies which were found in many important determinations, those of refraction and dispersion of optical glass for example, between the results of different experimenters. He was struck by the expedient of using a fibre or wire at the slit, and was surprised that this method had not been employed before. Some interesting stereoscopic effects ought to be observed if two differently coloured objects were viewed through pinholes in front of each eye. If the pinholes were further apart than the centres of the pupils, the distance between the blue image in one eye and that in the other would be less than the distance between the red ones, and the blue object would appear further off. If the pinholes were closer than the centres of the pupils, the reverse would be the casé, and the blue object would appear to be closer. In many opticians' windows the word FRIEND was exhibited, the letters F I and N being blue, while R, E and D were coloured red. On looking at this the red letters appeared to stand in front of the others. Prof. Von Rohr had once told him this was a true stereoscopic effect, the images on the two retinæ of a blue letter being closer together than the images of a red letter on account of the aberrations of the eye.

He had not given much thought to it himself, and would be interested if the author could explain the effect.

Mr. T. H. Blakesley said that if one formed a spectrum of a white light by a prism with refracting edge perpendicular to the slit, a drawn out image of the latter was obtained, which was white at the middle and coloured at the extremities. If this were now viewed through a pinhole which was moved to one side, he supposed the blue and red extremities would be deflected in opposite directions, while the central portion would be drawn out into a continuous spectrum. He did not think the refractive properties of the eye were sufficiently well known to justify actual numerical calculations of the Gaussian system.

Mr. S. D. Chalmers believed there had been a number of accurate determinations of the chromatic aberration of the eye. He was in the habit of assuming an aberration of $\frac{3}{4}$ Diopter (from C to F) in designing eyepieces, and for other purposes. He was interested in the author's wide slit method. He had himself found it better than using a fine slit to employ one of appreciable width and bisect it with the crosslines. Why not use a thick vertical wire in the eyepiece and a slit of three times the width. The setting obtained by bisecting the broad line with the wire was a very accurate one.

Mr. T. Smith pointed out that the result of § 5—viz., that the total parallax obtainable with a small fixed stop in front of the eye was independent of the positions of the objects—could be shown very simply. Owing to the stop the mean ray from any object was refracted at the same point of the refracting surface (approximately). The deviation of a blue ray incident at this point is independent of the angle of incidence, and $=\beta h$, where h is the distance from the axis, and $\beta=(\mu_b-1)/\mu_b r$, r being the radius of the surface. Similarly, the deviation of a red ray incident at the same point is ρh, where $\rho=(\mu_r-1)/\mu_r r$. The difference between the deviations of the two rays $=(\beta-\rho)h$, and is the parallax when the stop is at a distance h from the axis. Since β and ρ are independent of the inclinations of the incident rays, the parallax is independent of the distances of the objects from the eye. Mr. Guild's work in connection with these phenomena and their elimination in optical measurements had had important results. The precautions which he suggested had now been in operation at the laboratory for a considerable time, and the departmental records showed that the consistency of certain important determinations had been increased by at least five times. It was obviously essential to remove chromatic parallax if only because the errors due to it were so unsystematic.

Lieut.-Col. J. W. Gifford (communicated remarks) asked if the author had worked with an ordinary Ramsden eyepiece and shadow pointer—that is, a knife-edge pointer reversed so that there is no reflected light from the pointer. With shutters added so as to block out everything but the line under measurement he had found such an arrangement more accurate than any other he had used. Of course, the projected shadow of the point must be accurately focussed by the eyepiece for each line to be measured.

The Author replied as follows: I was not aware that many accurate determinations of the chromatic aberration of the eye had been made as stated by Mr. Chalmers. P. G. Nutting states in his Paper that the matter had rested with the first rough determinations of Helmholtz until his own experiments. I have not seen any work on this subject since then. Mr. Chalmers's statement that he makes an approximate allowance for this aberration in designing eye-pieces is of great interest. I have myself used a slit of appreciable width, bisecting it with the cross lines (inclined type), but do not like the arrangement nearly so well as the central fibre method described in the Paper; since, if the slit is sufficiently wide to render the cross clearly visible and free from diffraction troubles, the setting is not

nearly so certain as that on a fine black line. With regard to the use of a thick vertical cross line and a slit of about three times its width, as suggested by Mr. Chalmers, this type of setting, in which the vertical wire is set so as to bisect the spectrum line, is hardly suitable for spectroscopic work for the following reasons: A symmetrical setting is certain to be estimated wrongly unless there is almost complete symmetry throughout the field of view. This is never the case in spectrum work unless there is only one line visible at a time. Further, the vertical wire is straight, whereas the spectrum lines are appreciably curved Thus the line can never be bisected for its whole length, and it is difficult to make a satisfactory bisection in the middle of the line when the condition does not hold for other parts of it. Any personal errors arising from this cause will come fully into the final result; for the zero reading to which the others are referred, being made on the unrefracted image of the slit, is not subject to this effect. A further objection, which applies to all wide slit methods, is the difficulty of close doublets. The sodium lines are involved in nearly all optical determinations, and it must be possible to deal with them under all degrees of dispersion from complete non-resolution to comparatively wide separation. The symmetrical setting of a vertical wire in a band of appreciable width is clearly bound to be influenced by the proximity of the adjacent bright band, even if the dispersion is sufficient to prevent overlap. If they overlap the setting is probably even worse. I have not seen the type of shadow pointer mentioned by Mr. Gifford, and so cannot speak of it from personal experience. From the description of it I conclude that it also is a wide slit method, and is therefore liable to some of the objections mentioned above when dealing with close lines. Further, the necessity of blocking out the rest of the field appears to be somewhat objectionable, on the ground that every extra adjustment which has to be made, especially near the end of the telescope or collimator, is a possible cause of error.

The spider line method advocated in section 8 is not, of course, a wide slit method in the sense of those mentioned above. Essentially, even the setting of the intersection of diagonal cross lines on a fine line, whether bright or black, is a symmetrical setting; but the area on which attention is concentrated is so much smaller than in the other types of setting considered that the disturbing effect of an adjacent line is practically inappreciable. In the foregoing it may seem as if the importance of small effects is overestimated; but it cannot be too strongly emphasised that the present state of optical science is such that any further researches on refraction and dispersion must, if they are to be valuable, attain an accuracy corresponding to about a second of arc in the angular measurements. This can never be attained if any possible source of error remains either in the design of the instrument or in the methods of using it. Prof. Cheshire's* suggestions are of considerable interest. The stereoscopic effect due to pin holes displaced in opposite directions in front of the two eyes can readily be observed A convenient arrangement is to mount the pin holes on the trial spectacles used by optometrists, when the distance between them can easily be adjusted to any desired value. If a number of rectangular slots cut in black paper and backed by alternate red and blue filters are used as objects, the effect is well marked, provided care is taken to see with both eyes, and not only with one.

I have not seen the actual optician's sign in which the word "Friend" in red and blue letters occurs. A hasty trial experiment, made before the meeting with the alternate red and blue slots mentioned above, convinced me that the red appeared to stand out from the blue quite as much when seen with only one eye as with both, and that the effect was monocular and not stereoscopic. This view appeared to be shared by Mr. Chalmers. Afterwards various observers were asked to come separately into the room in

* The reply to Mr. Cheshire is based on experiments performed afterwards.

which the coloured slots were mounted, with one eye closed and with no knowledge of what they were to look for. In every case the red was said to appear appreciably nearer than the blue, and in no case was the effect increased by using both eyes. Thus there is undoubtedly a monocular effect, which is simply due to the fact that the images of the blue slots are formed in front of the red. The mind is accustomed to judge (from the necessary effort of accommodation) that if there are two images at different distances within the eye the anterior one is due to a more distant object, and so the blue slots are assumed to be further away than the red. Although no true stereoscopic effect was detected with these coloured slots, some further experiments appeared to show that there is a binocular effect in addition to the monocular one; and this arises from the fact that the line of sight is not coincident with the axis of the eye, but is inclined inwards with respect to it. The aperture of the incident beam, when the whole eye is used, is limited by the pupil, or rather by its virtual image in the refracting surface of the cornea; and the pupil is concentric with the axis.* The line of sight—which passes through the front nodal point—does not, therefore, pass through the centre of the pupil. The virtual pupil is roughly about 5 mm. anterior to the nodal point. The line of sight is inclined to the axis at about 6 deg., it therefore passes the virtual pupil at about 0·5 mm. from the centre, measured towards the nose. Thus the mean ray of the incident pencil, which passes through the centre of the pupil, is incident about 0·5 mm. to the outside of the line of sight; and is in consequence deflected inwards, a blue ray being more bent than a red. The difference in deviation from C to G' can be obtained by substituting 0·5 mm. for h in the formula for parallax in section 5; it is about two minutes, but will vary considerably in different eyes, since errors in the relative centering of the various parts of the eye amounting in some cases to 0·25 mm. are frequent. But, in general, blue images are formed slightly more to the nazal side of the retina than red ones (the objects being equally distant). Less convergence is therefore required to fuse the blue images when both eyes are employed, and so the blue object appears further away.

The difference in deviation of blue and red rays coming from an object on the line of sight was experimentally verified by stretching a thin wire vertically in front of a red and blue filter, mounted one above the other and backed by illuminated ground glass. This was placed about three-quarters of a metre away, so that both halves of the wire were seen equally sharp. When looked at with one eye a distinct discontinuity of the wire at the junction of the filters was observed. If the right eye was used, the part of the wire on the blue field appeared displaced to the right; while with the left eye it appeared displaced to the left, each case corresponding to the greater nazal deviation of the blue rays. With both eyes open the stereoscopic effect was distinctly visible. These effects were also observed by persons with no foreknowledge of the possible results. It is clear that an error from this cause would result if one attempted to set red cross lines on a blue line, using the whole pupil of the eye, or to set black cross wires illuminated by red or white light on a blue line. No parallax would be observed, for movements of the eye would not alter the displacement of the mean ray from the line of sight; but a constant error would result

This source of error will affect observations made with low power instruments, in which the whole pupil is filled, if the cross lines are improperly illuminated. It is possible that this point has not been given sufficient attention in the design of low-power telescopes for use by night. Frequently the cross lines of such instruments are illuminated with a dull red light. The objects sighted are probably never so blue in colour as to introduce appreciable error from the cause discussed; but it would seem to be a safer practice to illuminate the cross lines with light from an intermediate part of the spectrum.

* Usually; but it is often found displaced outwards from the axis, which would increase the effect discussed.

XXIV. *The Capacity of an Inverted Cone and the Distribution of its Charge.* *By Prof.* G. W. O. Howe, *D.Sc.*

Received May 25, 1917.

The author has recently shown that the consideration of the equivalent capacity and inductance of a radio-telegraph aerial, consisting of a plain vertical wire, is simplified by assuming it to be an inverted cone.* In this connection it was considered desirable to calculate the ordinary electrostatic capacity of such a conductor, taking into account the effect of the earth with which the apex is almost in contact. So far as the author is aware this problem has not been previously solved.

The method employed is one which the author has already used on previous occasions for the calculation of the capacity of various types of aerials.† It consists essentially in considering the conductor as divided into a number of sections—the number depending on the accuracy required—each of which is subdivided into a large number of small elements, insulated from each other, and assuming that the charge is distributed uniformly over each section. Formulæ are then established for the average potential of each section, and the values of the densities on each section are so determined that this average potential is the same for each section, and, therefore, for the whole conductor.

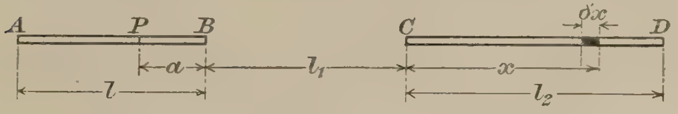

Fig. 1.

The effect of the earth is allowed for by the method of images. The only assumption made is that, when the charge has its natural non-uniform distribution, the potential does not differ appreciably from the calculated average value with the assumed artificial distribution of the charge. The accuracy of the method is limited solely by the number of sections into which the conductor is divided, and the consequent number of simultaneous equations which have to be solved.

* The Year-Book of Wireless Telegraphy, 1917, p. 694.

† Electrician, Vol. LXXIII., pp. 829, 859, 906; Vol. LXXV., p. 870; Vol. LXXVII., pp. 761, 880.

In the case of a cone it is advantageous to make a further simplification by replacing the conical sections by cylindrical ones of the same surface area, which, if the cone is of small angle, are approximately of the same mean diameter as the conical sections.

Since the average potential of any section is the sum of its average potentials due to its own charge and the charges on the other sections, it will be necessary to find the average potential of one conductor AB (Fig. 1) due to a uniformly distributed charge on another conductor CD in the same straight line.

If the conductor CD has unit charge per centimetre of length, the potential at P due to the element δx is $\delta x/(l_1+a+x)$, and that due to the whole conductor CD is

$$\int_0^{l_2} dx/(l_1+a+x)=\log(l_1+l_2+a)-\log(l_1+a)=V_p.$$

The average potential of AB due to the charge on CD is

$$\frac{1}{l}\int_0^l V_p da=\frac{1}{l}\Big\{(l+l_1+l_2)\log(l+l_1+l_2)-(l_1+l_2)\log(l_1+l_2)$$
$$-(l+l_1)\log(l+l_1)+l_1\log l_1\Big\}.$$

If $AB=CD=l$, this becomes

$$\frac{1}{l}\Big\{(2l+l_1)\log(2l+l_1)-2(l+l_1)\log(l+l_1)+l_1\log l_1\Big\}.$$

In our case the gap l_1 is made equal to o, l, $2l$, $3l$, &c , which leads to further simplification.

If $l_1=o$, the average potential $=2\log 2 \qquad =1{\cdot}3863$
$l_1=l$,, ,, ,, $=3\log 3-4\log 2=0{\cdot}52323$
$l_1=2l$,, ,, ,, $=4\log 4-6\log 3$
$+2\log 2=0{\cdot}3398$
$l_1=3l$,, ,, ,, $=5\log 5-8\log 4$
$+3\log 3=0{\cdot}25271$
$l_1=4l$,, ,, ,, $=6\log 6-10\log 5$
$+4\log 4=0{\cdot}20132$

. . . .

$l_1=nl$,, ,, ,, $=(n+2)\log(n+2)$
$-2(n+1)\log(n+1)+n\log n.$

The average potential of a straight wire due to its own uniformly distributed charge has previously been shown to be equal to $2\log(l/r)-0{\cdot}6137$ for unit charge per centimetre.*

* Electrician, Vol. LXXIII., p. 830, 1914.

These formulæ can now be applied to the case of the inverted cone.

In order to illustrate the effect of the degree of subdivision, it will be assumed in the first place that the whole cone is replaced by a cylinder of the same mean radius ; the cone will then be subdivided into two sections, then into three and finally into four.

1. The whole cone replaced by a single cylindrical section (Fig. 2). Let h=height of the cone, r=mean radius of the cone. Assume a uniform charge of 1 unit per centimetre of

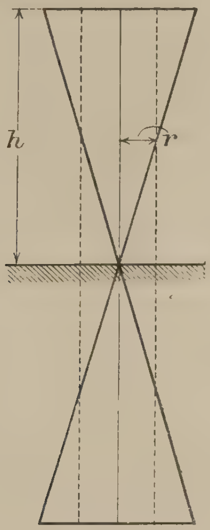

FIG. 2.

length, then the average potential due to its own charge is $2 \log (h/r) - 0{\cdot}6137$, whilst that due to the equal and opposite charge on the image is $-1{\cdot}3863$. Hence, the resultant potential is $2 \log (h/r) - 2$ and the capacity

$$=\frac{h}{2 \log (h/r) - 2} \text{ cm.}$$

If h=100 ft and r=2 mm. the capacity is 176·6 cm.

2. The cone divided into two sections, each being replaced by a cylinder of the same mean radius (Fig. 3)—

For the upper section, $l=h/2$, radius$=3r/2$, charge$=q$ units per centimetre of length.

For the lower section, $l=h/2$, radius$=r/2$, charge$=1$ unit per centimetre of length.

The average potential of the upper part due to—

its own charge	$=q(2\log(h/3r)-0{\cdot}6137)$,
the lower part	$=1{\cdot}3863$,
the image of the lower part	$=-0{\cdot}52323$,
the image of the upper part	$=-0{\cdot}3398\,q$,

giving a resultant average potential of—

$$(2\log(h/3)r-0{\cdot}9535)q+0{\cdot}8631.$$

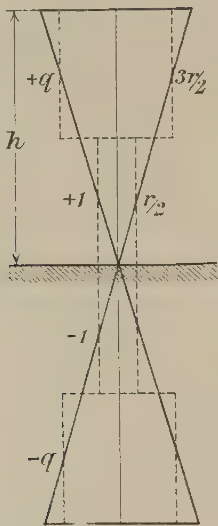

FIG. 3.

The average potential of the lower part due to—

its own charge	$=2\log(h/r)-0{\cdot}6137$,
the upper part	$=1{\cdot}3863\,q$,
the image of the lower part	$=-1{\cdot}3863$,
,, ,, upper part	$=-0{\cdot}52323\,q$,

giving a resultant average potential of

$$0{\cdot}86307q+2\log(h/r)-2.$$

On equating the average potentials of the two sections it is found that

$$q=\frac{2\log(h/r)-2{\cdot}8631}{2\log(h/r)-4{\cdot}01379}.$$

Hence, the upper section has the greater charge per unit length.

If, as before, $h=100$ ft. and $r=2$ mm.,

$$q=1{\cdot}075,$$

the average potential is 18·193, the total charge 3,163 units, and the capacity 174 cm.

3. The cone divided into three sections, each replaced by a cylinder of the same mean radius. The length of each section is $h/3$, their radii are $r/3$, r and $5r/3$, and the charges per centimetre are 1, q, and q_1 respectively.

The average potential of the upper section due to—

its own charge and image $=q_1(2\log(h/5r)-0{\cdot}6137-0{\cdot}20132)$,

the mid-section and its image $=q(1{\cdot}3863-0{\cdot}25271)$

the apex section and its image $=0{\cdot}52323-0{\cdot}3398$,

giving a resultant average potential of

$$q_1\,(2\log\,(h/5\,r)-0.81502)+1.13359\,q+0.18343.$$

Similarly, it is found that the average potential of the middle section is

$$q(2\log(h/3r)-0{\cdot}9535)+1{\cdot}13359q_1+0{\cdot}86307,$$

whilst that of the apex section is

$$2\log(h/r)-2+0{\cdot}86307q+0{\cdot}18343q_1.$$

On equating these three values of the average potential and substituting the values of h and r already assumed, it is found that

$q=1{\cdot}0055$, $q_1=1{\cdot}1171$, the average potential $=18{\cdot}3376$, the total charge 3,174 units and the capacity 173·2 cm.

4. The cone divided into four sections each replaced by a cylinder of the same mean radius. Except that they are more laborious, the calculations in this case are similar to those for the case just considered. If $h=100$ ft. and $r=2$ mm. the results obtained are as follows :—

Charge per centimetre of length on the apex section					=1.
,,	,,	,,	,, next	,,	=0·9936.
,,	,,	,,	,, next	,,	=1·031[illegible].
,,	,,	,,	,, top	,,	=1·1458.
Total charge					=3,180 units.
Average potential					=18·41.
Capacity					=172·75 cm.

The results are plotted in Fig. 4, from which it is seen that the capacity of the cone is about 171·5 cm. In the upper part of Fig. 4 the charge per centimetre of length in the case of the subdivision into four sections is plotted, and a curve

drawn through the mid-point of each section. This shows the approximate distribution of the charge over the actual cone. It will be seen that over the lower half of the cone the charge has practically a constant value per unit length, whilst above the mid-point the charge per unit length increases rapidly, and at the upper extremity is almost proportional to the radius; in

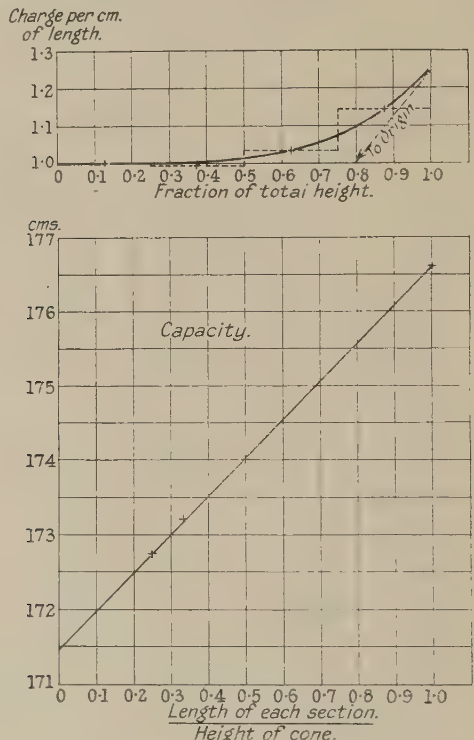

FIG. 4.

other words, the density over the lower half is inversely proportional to the distance from the ground, but above the mid-point it falls off much less rapidly, and is almost uniform near the upper extremity.

The capacity of a very acute inverted cone with its apex almost in contact with the ground is seen to be about 2·9 per cent. smaller than that of a cylinder of the same height, with a radius equal to the mean radius of the cone. This can be calculated by the formula

$$C=\frac{h}{2\log(h/r)-2}\,cm.$$

XXV. *On the Measurement of Small Inductances and on Power Losses in Condensers.* *By* ALBERT CAMPBELL, *B.A.* (*From the National Physical Laboratory.*)

RECEIVED JUNE 7, 1917.

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Sifter Method with damped Oscillations.

Introductory.

THE measurement of very small self inductances of circuits having only two terminals is a matter of no great difficulty, if the resistances are not high.

One of the simplest ways of doing this is by Heaviside's method,* in which comparison is made with a known mutual inductance.

It is easy to build a mutual inductometer of almost any desired lowness of range by using the device of stranding the windings of the coils as I have described in an earlier Paper. The lowest range of the instrument used in the measurements here described was from 0 to 1 microhenry, the scale being readable to 0·001 microhenry at the upper part. An instrument of this kind combined with a constant inductance rheostat (and ratio arm coils) is sufficient for dealing with two-terminal resistances. This bridge system unfortunately is not directly applicable to four-terminal resistances, in which the potential terminals are distinct from the other two. Venner † has shown, however, that the inductances of such resistances can be determined by the addition of the Kelvin double bridge device to the Heaviside bridge.

Other methods not of bridge type have been used. Orlich ‡

* Phil. Mag., p. 173, Vol. 23, Feb., 1887. When I described and elaborated this method (Proc. Phys. Soc., p. 69, Vol. 21, 1908) I was not aware that it was one of Heaviside's very extensive series.

† Bulletin of the Bureau of Standards, p. 559, Vol. 8, 1912.

‡ Zeitschrift für Instrumentkunde, p. 114, Vol. 25, 1905.

employed an electrostatic wattmeter, and his method was used later by E. and W. H. Wilson.*

More recently several ingenious methods, chiefly using two-phase currents, have been described. In Silsbee's Paper† there is an interesting discussion of "impure" mutual inductances, namely, those in which there is internal power loss (by eddy currents or dielectric hysteresis, &c.), and by a useful convention the time constant of such circuits is defined.

Methods Using M, R Element.

The methods which I proceed to describe are simple in their working, requiring only a single-phase source of current; on the other hand, they have certain limitations in practice, which will be discussed below. They are developments of the method of testing transformers which I described some years ago.‡ The fundamental principle in that method was the introduction of the elementary pair (M, R) shown in Fig. 1,

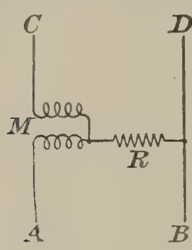

Fig. 1.

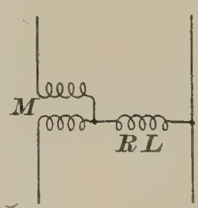

Fig. 2.

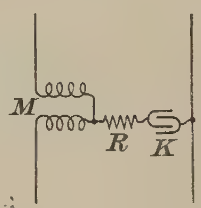

Fig. 3.

where A B are current terminals and C D potential points (or vice versa).

Other methods of utilizing this element have been developed by other observers,§ and it has formed the basis of the simplest alternating-current potentiometer, that of Prof. A. Larsen,‖ of Copenhagen.

* Electrician, p. 464, Vol. 56, June, 1906.

† Silsbee, Bulletin of the Bureau of Standards, p. 375, Vol. 13, 1916.

‡ Report of National Physical Laboratory, March, 1909, and Proc. Phys. Soc., p. 497, Vol. 22, 1910.

§ C. H. Sharp (in Discussion, June 30, 1909), Trans. American Inst. of Elec. Engs., p. 1040, Vol. 28 (2), 1910; Sharp and Crawford, Trans. A.I.E.E., p. 1517, Vol. 29 (2), 1911; Agnew & Silsbee, Trans. A.I.E.E., p. 1635, Vol. 31, 1912.

‖ ("Der Komplexe Kompensator") Electrotechnische Zeitschrift, p. 1039, Vol. 41, Oct., 1910; and Electrician, p. 738, Vol. 66, Feb. 17, 1911. A mistranslation ("Compensation apparatus" for "potentiometer") in the title obscured the meaning for most English readers.

The elementary pair (M, R) may be further developed by associating with R either self-inductance or capacity as in Figs. 2 and 3. These two cases require the same mathematical treatment, but they have different applications in practice.

I shall first discuss the case of self-inductance.

Method 1.—In Fig. 4 let r be a resistance with potential terminals, having self-inductance l to be found. It is connected, as shown, with a low-reading inductometer giving mutual inductance m, and a pair of inductive coils whose mutual inductance M can be varied. The second of these

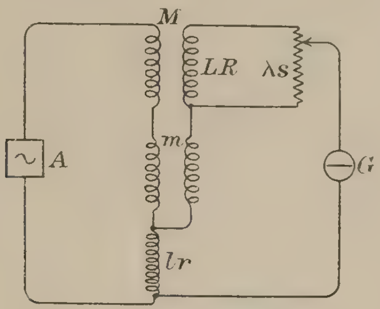

FIG. 4.

coils (of resistance R and self-inductance L) is connected to a resistance S having small self-inductance λ. (S may be adjustable by a slide-wire, in which case R and L must include the part above the slider.) A is a source of alternating current and G a vibration galvanometer or telephone.

If $\omega=2\pi n$, where n is the frequency of the source, it is easy to show that

$$MS=r(L+l)+(R+S)(\ +m), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$(L+\lambda)(l+m)\omega^2-\lambda M\omega^2=r(R+S). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, let L/R be (relatively) large, and λ/S small and λM negligible compared with $L\ (l+m)$; then equation (2) becomes

$$L(l+m)\omega^2=r(R+S), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives l conveniently without requiring an exact knowledge of M and λ.

In practice a disadvantage of this method is that usually the proper conditions are not easily obtained unless the frequency is tolerably high (say, 800 ~ per second). It should

be noted, however, that, as Bêthenod * and Orlich † have shown, the resistance and self-inductance of a properly designed shunt remain practically constant up to such frequencies.

Example.—The shunt to be tested had a resistance $r=0{\cdot}01$ ohm. With $R+S=5{\cdot}99$ ohms, $S=0{\cdot}274$ ohm, $L=0{\cdot}0100$ henry, λ less than 1 μ henry, $M \doteq 370$ μ henries, a balance was obtained when m was $0{\cdot}177$ μ henry, the frequency n being 800 ~ per second. Hence, from equation (3), $l=0{\cdot}063$ μ henry. The current used was of the order of 0·1 ampere, and was obtained from a small buzzer.

Method 2.—A method more generally applicable is shown in Fig. 5, r being the resistance whose self-inductance l is to be

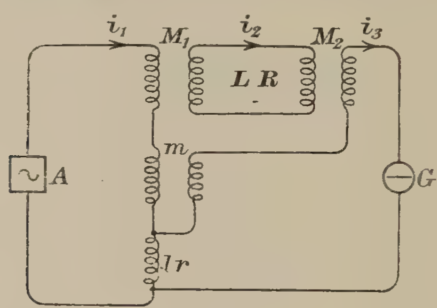

FIG. 5.

determined, and m a low-reading inductometer. The primary circuit is also linked to the galvanometer circuit by an intermediary closed circuit having resistance R and self-inductance L. The linking mutual inductances should be variable, but in general need not be known. There should be no direct mutual inductance between the primary circuit and the galvanometer circuit other than m. This condition can always be checked by opening the intermediary (L, R) circuit; then, when r is cut out m should read zero. The best plan is to put the four coils forming M_1 and M_2 at a good distance from the m inductometer, and to turn them so as to be conjugate to each other pair and to the inductometer coi's.

Let the instantaneous values of the currents in the three circuits be i_1, i_2 and i_3 as shown; and let $a=\omega\sqrt{-1}$.

* Jahrbuch der drahtlosen Tel., p. 397, Vol 2.
† Zeitschrift für Instrumentenkunde, p. 241, Vol. 29, 1909.

Then, when $i_3=0$, we have

$$(R+La)i_2+M_1ai_1=0,$$

and
$$[r+(l+m)a]i_1+M_2ai_2=0,$$

and hence

$$R(l+m)+rL=0, \qquad (4)$$

and
$$Rr=(l+m)L\omega^2-M_1M_2\omega^2. \qquad (5)$$

As mutual inductances are reversible we may take their values positive or negative in general Changing the signs of M_1 and m (and with $m>l$), we have

$$\frac{m-l}{r}=\frac{L}{R} \qquad (6)$$

and
$$Rr=[M_1M_2-(m-l)L]\omega^2. \qquad (7)$$

Equation (6) gives l without a knowledge of the values of M_1 and M_2.

From (6) and (7) we have

$$r=\frac{RM_1M_2\omega^2}{R^2+L^2\omega^2}\doteq\frac{M_1M_2\omega^2}{R},$$

when $L\omega/R$ is small,

and
$$m-l=\frac{LM_1M_2\omega^2}{R^2+L^2\omega^2}\doteq\frac{LN_1M_2\omega^2}{R^2},$$

where $L\omega/R$ is small.

The following two examples will give an idea of the relative magnitudes of the quantities involved for two different frequencies :—

r ohm.	l henry.	n ∾ per sec.	m/l.	R ohms.	micro-henries. L	M_1	M_2
0·01	0·05	100	2	10	50	250	1000
0·01	0·05	800	2	50	250	200	100

From equation (6) we see that $(m-l)$ must always be positive. From (6) and (7) we have

$$Rr=[M_1M_2-L^2r/R]\omega^2. \qquad (8)$$

Hence, $M_1\,M_1$ must always be greater than L^2r/R.

Current Transformer Compensated for Frequency.

The elementary pair (M, R) of Fig. 1 is of interest in connection with current transformers. In an ironless transformer let M be the mutual inductance, and let Q and L be the resistance and self-inductance of the secondary circuit. Then,

if I_1 and I_2 are the effective values of the primary and secondary currents, it is well known that

$$\frac{I_1^2}{I_2^2}=\frac{Q^2+L^2\omega^2}{M^2\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and accordingly the current transformation ratio is not constant for different frequencies.

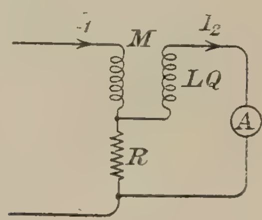

FIG. 6.

Now let a resistance R be inserted as shown in Fig. 6. Then we have

$$\frac{I_1^2}{I_2^2}=\frac{(R+Q)^2+L^2\omega^2}{R^2+M^2\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The ratio is independent of ω and equal to L/M, if

$$(R+Q)/R=L/M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

When this condition is satisfied the ratio of current transformation will be constant for all frequencies, and hence also for all wave forms.

Determination of Capacity in Terms of Mutual Inductance and Frequency.

In a former communication * I described a very simple method of measuring capacity in terms of mutual inductance and frequency. The method has proved very useful for quick and accurate determinations of frequency by the help of a known condenser and a variable mutual inductance.

With the method in its simplest form, as shown in Fig. 7, the mutual inductance m is adjusted until G, the vibration galvanometer (or telephone) shows no current. Then, $mk\omega^2=1$.

* Proc. Phys. Soc., p. 69, Vol. 21; and Phil. Mag., p. 155, Jan., 1908.

But unless the condenser is free from leakage (or other power loss), a perfect balance cannot be obtained, but only a minimum current in the detecting instrument. With an imperfect condenser, however, this trouble can be entirely eliminated by adding an auxiliary closed circuit similar to that

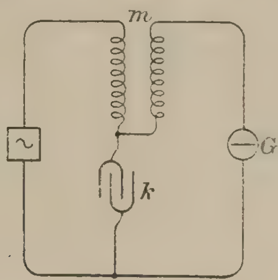

Fig. 7.

in Fig. 5. By this device the condenser losses can always be compensated for and a perfect balance obtained. The complete method is shown in Fig. 8, in which the imperfection of the condenser is represented by the series resistance r.

Let the current in G be reduced to zero by adjusting m and M_1 or M_2.

By writing $-1/k\omega^2$ for l in equations (6) and (7) and chang-

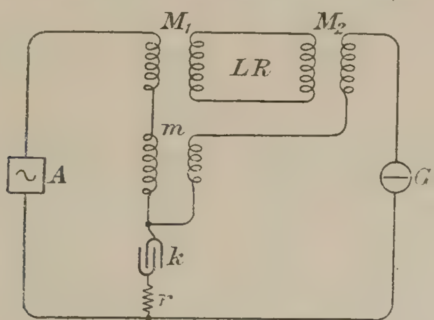

Fig. 8.

ing the sign of m the conditions for balance are obtained, namely

$$1/k\omega^2 = m + rL/R, \qquad (12)$$

and

$$Rr = [M_1 M_2 + (m - 1/k\omega^2)L]\omega^2. \qquad (13)$$

Hence

$$Rr = [M_1 M_2 - L^2 r/R]\omega^2. \qquad (14)$$

Here also M_1M_2 must be greater than L^2r/R, except in the limiting case of a perfect condenser, where $r=0$ and then M_1M_2 also $=0$.

From the above equations we have

$$r=RM_1M_2\omega^2/(R^2+L^2\omega^2), \quad . \quad . \quad . \quad . \quad (15)$$

and

$$1/k\omega^2=m+LM_1M_2\omega^2/(R^2+L^2\omega^2) \quad . \quad . \quad (16)$$

When ω is known, these equations give k and r and hence $rk\omega$, the power factor of the condenser.

When the method is used for measuring frequency, it is best to arrange the auxiliary closed circuit so that L/R is very small, and hence rL/R negligible compared with m.

Then

$$\omega^2 \doteq 1/km, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and the frequency is determined as though the condenser were perfect, and without a knowledge of the values of r, L, R, M_1 and M_2, the auxiliary circuit merely introducing a small vector which balances the ri_1 without perceptible effect in equation (17).

In general the auxiliary closed circuit can be a few turns of wire mounted so as to be movable close to the coils of the inductometer giving m. Sometimes even a coin shifted about above the coils is sufficient to make the balance exact. With a given condenser the scale of the inductometer can be graduated to read the frequency directly, and with a suitable series of condensers the same scale may be used with even multipliers.

When the wave form of the source of current contains strong harmonics these may somewhat obscure the point of balance when a telephone is used as the detecting instrument. For frequencies from 200 up to 5,000 ∞ per second, the harmonics cause very little trouble if the primary self-inductance in the inductometer is kept relatively high. [The system of Fig. 4 can also be used with a condenser in place of l.]

In a former communication * I have described the use of the condenser and mutual inductance combination as a wave sifter by which a component of any desired frequency can be suppressed in a given circuit. The addition of the auxiliary closed circuit here described makes the sifting perfect even when quite common paraffin paper condensers are used.

When the power loss in the condenser (and hence its power factor) is to be determined, usually it is best to keep L/R very small.

Then

$$r \doteq M_1M_2\omega^2/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

* Proc. Phys. Soc., p. 107, Vol. 24, 1912.

In this case L need not be known accurately, but $M_1 M_2$ must be determined. The same care must be taken with regard to the position of the coils forming M_1 and M_2 as in Method 2.

Or $M_1 M_2$ may be directly determined by adding a known resistance s to the condenser circuit and altering R to R' to obtain a rebalance.

Then $$M_1 M_2 \omega^2 = s/(1/R' - 1/R). \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In this case the loop (L, R) may be placed over the inductometer coils if desired.

The following example gives an idea of the relative values of the various inductances and resistances in the determination of the power factor of a good quality mica condenser of capacitance 0·1 μF and power factor 0·0005 using a frequency of 800 ∞ per second.

Here $\omega \doteq 5000$, $m = 0{\cdot}4$ henry, and $r = 1$ ohm.

If $R = 10$ ohms, then $M_1 M_2 = 4 \times 10^{-6}$, and M_1 and M_2 can be 2 millihenries each.

It should be remarked, however, that, altho' this method of determining the power factor of a condenser may be a good one for certain cases, for general use the Carey Foster method appears to hold the highest place for simplicity and accuracy.

Sifter Method with Damped Oscillations.

If the source gives damped oscillations of the form $\varepsilon^{-bt} \cos \omega t$, where $b = n \times$ log. decrement, then (Fig. 7) it is easy to show that, for a balance,

$$mk(\omega^2 + b^2) = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and $$2mb = r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where r is the series resistance including that which represents the loss in the condenser. Here no auxiliary circuit is required to balance the loss, if by external addition r can be made to satisfy equation (21).

In conclusion, I would express my thanks to Sir Richard Glazebrook, F.R.S., for kind interest in the work.

XXVI.—*The Mechanism of Colour Vision.* By J. GUILD, A.R.C.S., D.I.C., F.R.A.S.

RECEIVED JULY 2, 1917.

IN a recent Paper* to the Royal Society, Dr. R. A. Houstoun has outlined a new theory of the mechanism of colour vision, in which the three sets of resonators of the Young-Helmholtz theory are replaced by a single set, of which the resonance point corresponds to $\lambda=0{\cdot}55\mu$.

The present Paper is primarily a criticism of the suggested theory and of the premises from which many of its main features are deduced.

The theory supposes the existence in the eye of a great number of vibrators with a free period in the green, and it is assumed that the motion of one of these may be represented by the equation

$$\frac{d^2x}{dt^2}+h\frac{dx}{dt}+n^2x=E\cos\omega t, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

x being the displacement from the position of rest at time t, and $E\cos\omega t$ the force per unit mass exerted on it by the incident light wave. Starting from this, it is deduced that the ratio of the absorbed to the incident energy of any wave-length, λ, is proportional to the quantity,

$$\frac{\lambda^2}{(\lambda^2-a^2)^2+b^2\lambda^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is the resonance wave-length and b is another constant. It is then further assumed that the luminosity of the light is simply proportional to the energy absorbed by the resonators, and that the above expression, therefore, represents also the ratio of the luminosity to the incident energy, *i.e.*, the visibility of the radiation of wave-length, λ.

Suitable values of a and b being chosen, the expression is plotted in comparison with the experimental visibility curve.

There are several assumptions underlying this treatment which do not appear to the present writer to be justified. In the first place, without more knowledge than we at present possess concerning the nature and properties of the resonators in question, the most general significance must be given to the

* "A Theory of Colour Vision," Proc. Royal Soc., Vol. 92, p. 424.

quantities occurring in the oscillation equations; and to adopt values for the constants a and b which give the closest agreement with an experimental curve seems a som what arbitrary proceeding. Moreover, it is quite possible—if it is not, indeed, probable—that the behaviour of the oscillators cannot be represented by such a simple expression; h or n, or both, may not remain constant over the range of displacements and rates of change of displacement occurring in vision: in which case neither a nor b in the above expression for the ratio of absorbed to incident energy can be regarded as constant. It might easily be that the oscillations depart so far from these conditions that the form of the expression obtained for the absorbed energy is quite misleading.

The second assumption, that the luminosity is proportional to the amount of energy absorbed by the resonating system, *whatever the wave-length*, appears not only to be devoid of justification, but to be very improbable. It has to be borne in mind that we are not here comparing physical effects with their appropriate phys cal causes: lum nosity is a measure of the *sensation* produced—*i.e.*, it is a mental, not a physical, phenomenon; and in passing from the physical processes occurring at the retina to the sensations to which they give rise in the brain we have at some stage to cross the unknown bridge between matter and thought. Knowing, as we do, nothing whatever of the process whereby energy, functioning on the physical plane as vibrations of a sense organ, is made to function on the mental plane as the corresponding sensation, it is obviously impossible to take for granted, as Dr. Houstoun does, that the conversion factor, if we may employ the term, from stimulus to sensation is independent of the frequency of the former. Even before this stage is reached, the energy absorbed by the resonators has to pass to the brain through a complex system of tissue layers and nerve fibrils. To assume without evidence that impulses of all frequencies are transmitted with equal efficiency by such a system can hardly be justified.

Thus, from the considerations outlined above, it would appear that:—

(*a*) There is some reason for doubting that the expression obtained by Dr. Houstoun from the properties of a simple type of resonator really represents the ratio of absorbed to incident energy for wave-length λ in the case of the eye.

(*b*) Even were this ratio known with certainty, there is no

justification for regarding it as a measure of the relative visibility of radiation of different wave-lengths.

The agreement of Dr. Houstoun's curve with the experimental curve strikes the writer as very superficial, and seems to prove nothing beyond the fact that any mathematical curve which has a peak in it will, if plotted to a suitable scale, with its constants suitably chosen, bear a considerable resemblance to any other peaked curve.

Fechner's Law.

This law, which is sometimes referred to as Weber's, states that the smallest increment of intensity which the eye can detect bears a constant ratio to the total intensity.

In order to make his expression for visibility fit in with this law, Dr. Houstoun evolves a scheme whereby an ever-increasing number of resonators is thrown out of action as the intensity of the incident light increases.

This appears to be wholly arbitrary and unnecessary. The established fact is that dI/I=constant, where dI is the minimum difference in intensity which the eye can detect when the total intensity is I. Dr. Houstoun then assumes that $dS \propto dI/I$, where dS is the increment of sensation produced by an increment, dI, in the incident intensity. But the increment of energy absorbed by *each* resonator is proportional to dI, since expression (2) is constant for any given wave-length, and this necessitates a progressive diminution in the number of active resonators, as the intensity increases, in order to account for the lower rate of increase of the sensation. Reverting to the relation $dS \propto dI/I$, it follows from the fact that the smallest detectable value of dI/I is constant that the smallest detectable value of dS is also constant—*i.e.*, the relation $dS \propto dI/I$, with all the consequences which it involves, is based on the assumption that the smallest increment in the *sensation* of light which the mind will notice is constant, whatever the magnitude of the sensation to which it is added. As well might we expect that because the addition of an extra inch to a foot rule would be detected readily, the addition of an inch to a mile would also be noticeable.

This is only another example of the danger already mentioned of leaving out of account the unknown process by which physical causes produce mental effects, and of assuming that all peculiarities in sensation must necessarily be due to peculiarities of the sense-organ concerned.

In the view of the writer, Fechner's law is psychological, and not physiological, in origin; it is merely a particular case of a general law which applies to all five senses, and which may be expressed by saying that the mind takes cognisance of *relative* magnitudes only. The application of this law to some of the other senses is a matter of everyday observation; and is so very well known that nobody troubles to notice it, still less to represent it in terms of differentials or logarithms. For example, we have no difficulty in telling which of two weights is the heavier where one weighs half an ounce and the other a quarter, but it is a very different matter if one is fifty pounds and the other fifty pounds and a quarter of an ounce. As a matter of fact, the minimum difference which we can detect between two weights from their feeling of heaviness is very roughly proportional to their magnitude, yet no one would suggest that, with increasing load, there is a progressive diminution in the number of active nerve centres in the muscles of the hand or arm to account for our inability to detect the additional weight when a quarter of an ounce is added to fifty pounds.

Many similar examples could be quoted in connection with other senses. For instance, a few grains of common salt dissolved in a glass of water is obvious to the sense of taste, but the same few grains added to water already moderately saline would pass unnoticed.

In a quiet room the ticking of a clock or the slightest whisper is distinctly audible, but in a busy street or workshop we may have to shout in order to be heard. Yet we do not find that part of the mechanism of the ear goes out of action as the volume of sound increases. In fact, we know that, very approximately, the disturbances of the tympanum (and other parts of the auditory apparatus), due to different sounds, are superposed, and that the amount of energy absorbed by the ear from a particular sound is approximately independent of whether it is the only sound heard or one of many. Nevertheless, whether we are conscious of hearing it or not depends on its relative importance to the total amount of sound-sensation which the mind is perceiving at the moment.

Examples of this kind could be multiplied almost indefinitely, showing that whenever the mind deals with magnitude it is *relative* magnitude only with which it concerns itself; and a given increment in any magnitude will be perceived only if it produces a sufficiently important alteration in the total magnitude.

Enough has probably been said, however, to make it clear that Fechner's law has not to be explained by retinal peculiarities, as attempted by Dr. Houstoun; and that it does not, as he supposes, involve (for a given wave-length) the non-proportionality of sensation to intensity of incident light. On the contrary, the writer believes that there is at least presumptive evidence for regarding dS/S=constant as a fundamental psychological law connecting the magnitude of a *sensation* with the smallest change in that *sensation* which the mind will detect; and that, therefore, where we find a similar law—such as Fechner's—holding between the minimum detectable change in the *external impulse*, and the total magnitude of that *impulse*, such external law proves, within the limits in which and to the order of accuracy with which it holds, that the sensation is proportional to the physical impulse producing it. Thus Fechner's law simply shows that dI/I=constant=dS/S; and that, consequently, S is proportional to I. There is, therefore, no necessity for assuming a diminution in the number of active resonators at high intensities.

According to this view, it is not Fechner's law that requires a physiological explanation, but rather the fact that it only holds approximately; but when one considers the nature and complexity of the mechanism by which we rob the light wave of its energy and convey it to the brain it is surprising that Fechner's law holds even as well as it does.

After disposing of Fechner's law by ascribing to his resonators the very arbitrary properties already mentioned, Dr. Houstoun discusses the Purkinje effect. Rejecting the hypothesis that this is due to the progressive change from rod to cone vision as the intensity increases, he advances two possible explanations to explain the effect with one system of vibrators only.

The first of these assumes the vibrators to be embedded in a yellow medium, and would, therefore, confine the phenomenon to the region of the macula lutea; since, outside this small patch, no such yellow colouring matter exists. This is contrary to experience, since it is outside the yellow spot that the effect is most marked. In fact, in the centre of the fovea, where there is plenty of yellow colouring (*but no rods, and therefore no possible change from rod to cone vision*), the Purkinje effect is said to be absent altogether.

The second explanation—to which the author of the theory

himself inclines—assumes that the resonators are not all of one frequency, but have frequencies grouped about a mean. There is nothing inherently improbable in this assumption ; but the explanation also assumes the diminution, at higher intensities, of the effective number of resonators, an assumption which has already been criticised as being based on an erroneous conception of the origin of Fechner's law.

In rejecting the usual explanation of the Purkinje effect, Dr. Houstoun seems to leave out of account many undisputed facts connected with it. He does not think it necessary to " assume" two different mechanisms ; but the existence of the rods and cones, and the differences between them, are not matters of assumption, but well-known physiological facts. Further, the fact that the sensation of light persists in those portions of the retina where there are plenty of rods, down to far lower intensities than within the fovea, where there are no rods, if it does not absolutely prove the ordinary view that the rods are chiefly responsible for vision at low and the cones at high intensities, at least makes it a much more likely and legitimate assumption than any of those with which Dr. Houstoun attempts to replace it.

There is still another point at which the theory joins issue with experimental fact. We may profitably quote from the original : " But, it will be asked, how does this theory explain the apparent trichromatism of our ordinary sensations ? We must here fall back on the reason give by Dr. Edridge-Green—namely, that the colour-perceiving centre in the brain is not sufficiently developed to discriminate between the character of adjacent curves. Two curves must be widely different in shape and position before the colour-perceiving centre can detect the difference. A curve has an infinite number of points on it ; the colour-perceiving centre is so badly developed that, as far as it is concerned, the curve is sufficiently specified by three points on it, provided that these points are distributed over the spectrum. We can, therefore, represent our energy curve by three points."

If this view were correct, a given colour could be matched sufficiently well to satisfy our " badly-developed " colour perception with suitable proportions of any three colours so long as these were " distributed over the spectrum." Certainly it should be possible to select the red, green or violet constituents from within a fairly wide range.

Experiment has abundantly shown, however, that unless

the three constituent colours are chosen of wave-lengths lying within very narrow limits our colour perception is so *well* developed that it is impossible to reproduce other colours to our satisfaction. The three primary wave-lengths of the trichromatic theory cannot, therefore, be regarded as three points on a single curve with no further significance than that they are reasonably spaced over the spectrum. Experiment has fixed them for us within much narrower limits than is consistent with this view.

Dr. Houstoun claims the fact that the colour of monochromatic light becomes whiter at high intensities as supporting his theory; but this phenomenon is also accounted for with perfect ease on the trichromatic theory, and cannot, therefore, be cited as a discriminant.

In conclusion, it may be pointed out that the proposed theory, even if the validity of all the assumptions which the writer has criticised be allowed, does not do more than explain in an approximate manner the general characteristics of *normal* colour vision. The author makes no attempt to account for the multitude of phenomena of complete and partial colour blindness which have served to establish the trichromatic theory both qualitatively and quantitatively. The only reference to colour blindness is contained in a more popular résumé of the theory given in Science Progress,* in which the author says : " Instead of accounting for colour blindness by a want of discriminating power in the colour-perceiving centre of the brain, we may ascribe it to excessive disturbance of the vibrators in the retina. This would make the energy curves widen out and lose their distinctive form." It is scarcely necessary to point out that colour blindness is not accounted for on the tri-chromatic theory by want of discriminating power in the brain ; but is supposed to be due to one of the three sets of resonators being wholly or partially out of action. The difficulty of a red-blind man with a deep red light is not that he cannot tell its colour, but that he cannot see it. How excessive disturbance of the vibrators should result in absence of sensation in such cases is difficult to follow. On this theory the brightness of a white light should be greater in the case of a man who is wholly or partially colour blind on account of this excessive disturbance of the vibrators ; whereas, as is definitely established, it is less.

* No. 43, January, 1917, p. 387.

After all, the objection which Dr. Houstoun has to the trichromatic theory appears to be summed up in this: There is no physiological evidence of the existence of three sets of vibrators. Why should there be? It is very improbable that the rods and cones are themselves the resonators in question. It is much more likely that the resonators consist of specialised cells of ultra-microscopic—possibly molecular—dimensions within the substance of the rods and cones; in which case there is no physiological evidence of the existence of any of them. As a matter of fact, no one would associate anything in the microscopic structure of the retina with resonating systems at all. The evidence for the existence of any such system is wholly indirect: they have to be assumed to explain the known physical properties of the eye. If these properties include, as they must, the accumulated facts of colour blindness, it is difficult to see how the assumption of three sets of resonators is to be avoided. There may be no direct evidence of their existence; but there is plenty of evidence of the necessity for their existence.

XXVII.—*The Use of Monochromatic Interference Rings for the Measurement of Curvature.* By S. D. Chalmers, *M.A., Technical Optics Department, Northampton Polytechnic Institute.*

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Shallow convex surfaces can be compared directly with a plane surface. A measuring microscope is provided with a vertical illuminator arranged to direct monochromatic light on the stage; the convex surface is placed on the plane and the whole carefully centred. If necessary, the edges of the lens are supported to prevent rocking. The rings are focussed and carefully centred, and the diameters of convenient rings measured, say, the 3rd and 20th rings. Let the measured diameters be $2C_1$ and $2C_2$; then the radius of curvature of the plate is given by

$$r_1=\frac{C_2^2-C_1^2}{(20-3)\lambda}.$$

The difference $20-3=17$ is chosen because $17\times0{\cdot}0005894$ is approximately 0·01 mm., and the value of the radius is readily calculated as accurately as the measurements permit.

It has been assumed that the upper surface has no effect, but this can be readily calculated and allowed for. The difficulty, can, however, be avoided by placing the flat plate on the lens to be tested, supporting it on the edge by a thin layer of soft wax and pressing it down to give a central contact. If care be taken to keep the upper surface approximately level no difficulties arise.

Comparison of Two Curved Surfaces.

The same method can be employed to find the difference of curvature for the case of a convex and a concave curve giving central contact.

The procedure is as before, except that the difference of curvature,

$$R_1-R_2=\frac{(n_2-n_1)\lambda}{C_2^2-C_1^2},$$

where n_2 and n_1 are the numbers of the rings chosen. The number of rings chosen depends on the conditions of visibility,

but, as before, it is often convenient to make a measurement for a difference of 17 rings to facilitate the calculation.

The apparatus used consists of a 1″ o.g. on a fairly good measuring microscope, and with sodium light the method is applicable to a difference of curvature of about $\frac{1}{400}$ mm., and gives an accuracy of about 1 in 1,000 in this case.*

As the actual curvature increases the relative accuracy in measuring the difference becomes less important, and greater differences can be measured.

When the difference of curvature is small, so that the rings become very wide, it is preferable to remove the lenses and substitute a pin hole and a cross line.

The method involves the use of standard surfaces, but in the case of steep concave curves the comparison can be made with steel balls, which are commercially procurable, and are fairly accurately spherical. Such concave surfaces can be used to check other convex surfaces. The method described above applies only to the case of central contacts. In the case of edge contacts the following method may be used :—

The standard plates are smaller than the test surface, and are accurate up to the edge (in practice they are reduced after being worked), and the method used is to determine the central separation of the two surfaces.

The difference of the optical path $n\lambda$ for the two interfering beams at the centre will depend on the angle of incidence, and as the angle is increased the path difference will change, and rings will appear to run into the centre. We count the number of rings (p) which disappear into the centre while the angle of incidence changes from i_1 to i_2.

Then

$$p=(n \cos i_1 - n \cos i_2)\text{—}i.e., n\lambda=\frac{p}{\cos i_1-\cos i_2}\lambda$$

and the curvature difference$=\dfrac{p\lambda}{\cos i_1-\cos i_2}\Big/C^2$,

where C is the semi-diameter of the standard plate.

The apparatus used for the preliminary measurements consists of an open sight directed towards the axis of a circle and arranged so that its angular movement can be read on a circle (or a fixed angular movement given), and a table which

* In this case it is necessary to use the exact value of λ.

is adjustable to allow the rings to come to the centre of a circle.

The standard plate is placed on the surface to be tested, and the monochromatic rings produced by a sodium flame or mercury lamp. The sight is adjusted approximately to normal incidence, and a small glass plate used to illuminate the ring system. The central ring is brought on to the sighting line; the sight is then set at a fixed angle, say 30 deg., and the rings observed; if they are not on the sighting line the table is raised or lowered. The angle of observation is then slowly increased till the angle is, say, 60 deg., the number of rings which disappear being counted. It is easy to estimate to a quarter of one ring, so that the central path difference can be found to within one wave-length.

When the number of rings becomes excessive, intermediate values of i may be used to shorten the counting.

In order to keep the illumination uniform, a finely-ground glass plate is inserted between the flame and the surfaces.

The aperture of the open sight is preferably about 1 mm. in diameter, though a slot of less width is frequently more useful, especially when the path difference is large.

Limits and Precautions.

With sodium light the double line furnishes a limit at about 500 rings on the surface—that is, the separation must not exceed about 1 mm. If necessary, smaller test surfaces may be used, but with more monochromatic sources of light this limitation is not so necessary.

When the path difference is large a small aperture must be used in the sighting device, as otherwise the angles of incidence for the various rays reaching the pupil of the eye are different, and the interference pattern becomes indistinct.

If the separation of the surfaces at the centre be h, the path difference at the centre exceeds that at the edge by $2h \cos i$, where i is the angle of incidence of the light used for the observation.

In making an observation the light used will have slightly varying angles of incidence, and the path difference will vary with the angle of incidence, so that the change di in the angle of incidence will give a change $2h \sin i di$ in the path difference, or $n\lambda \sin i di$ when $n\lambda$ is the path difference for normal incidence.

The pattern will become indistinct when $n\lambda \sin i di$ exceeds $\lambda/2$, so that di should not exceed $\frac{1}{2 \sin i} \cdot \frac{1}{n}$. If $n=400$ and $i=45$ deg. di should not exceed $\frac{1}{560}$—*i.e.*, a slit $\frac{1}{2}$ mm. wide at 280 mm. distance from the pattern would suffice.

For determination of absolute curvatures it is desirable to keep the value of i_2 sufficiently large, so that the actual number of rings can be found. It is easy to estimate the number of rings which disappear to $\frac{1}{4}$ of a ring, so that where possible $\frac{1}{4}$ of the total rings should be counted, and $\cos i_1 - \cos i_2$ should equal $\frac{1}{4}$.

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